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V. S. Vladimirov

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Abstract

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MATHEMATICS

V. S. VLADIMIROV

ON THE CONSTRUCTION OF ENVELOPES OF HOLOMORPHY FOR DOMAINS OF A SPECIAL TYPE

(Presented by Academician N. N. Bogolyubov, 3 V 1960)

We shall denote by R^{n+1} the real space of dimension $n + 1$, and by $x = (x_0, x_1, \dots, x_n) = (x_0, \mathbf{x})$, $p, q, \dots$ its points; by $C^{n+1} = R^{n+1} + iR^{n+1}$ the complex space consisting of points $\zeta = (\zeta_0, \vec{\zeta}) = p + iq$; the scalar products $\zeta x = \zeta_0 x_0 - \vec{\zeta} \mathbf{x}$, $\vec{\zeta} \mathbf{x} = \sum_{1 \leq j \leq n} \zeta_j x_j$, $x^2 = x_0^2 - \mathbf{x}^2$; the Euclidean distances $|\mathbf{x}| = \sqrt{\mathbf{x}^2}$, $|x| = \sqrt{x_0^2 + \mathbf{x}^2}$. The future ($x_0 > |\mathbf{x}|$) and past ($x_0 < -|\mathbf{x}|$) light cones will be denoted respectively by Γ^+ and Γ^- , $\Gamma = \Gamma^+ \cup \Gamma^-$. The tube domains in C^{n+1} of the form $R^{n+1} + i\Gamma^+$ and $R^{n+1} + i\Gamma^-$ will be denoted respectively by T^+ and T^- , $T = T^+ \cup T^-$.

A smooth curve will be called **timelike** if its tangent vector belongs to Γ ; a smooth surface will be called **spacelike** if its normal belongs to Γ .

By a generalized function we mean any linear continuous functional on the Schwartz space S ((1), Ch. VII). The space of generalized functions (conjugate to S) will be denoted by S^* . Generalized functions $F^+(p)$ and $F^-(p)$ will be called **retarded** and **advanced**, respectively, if their Fourier transforms

$$\tilde{F}^\pm(x) = \int F^\pm(p) e^{-ipx} dp, \quad px = p_0 x_0 - \mathbf{p} \mathbf{x}$$

vanish respectively outside $\bar{\Gamma}^+$ and $\bar{\Gamma}^-$. A generalized function $f(p)$ will be called a **commutator** if $\tilde{f}(x) = 0$ for $x^2 < 0$.

We shall say that a function $F(\zeta)$, holomorphic in the domain T^+ (respectively T^-), belongs to the class N^+ (respectively N^-) if:

- 1) for any $q \in \Gamma^+$ (respectively Γ^-), $\varepsilon > 0$, and $t \geq t_0 > 0$, the estimate

$$|F(p + itq)| < c(q, \varepsilon, t_0) e^{\varepsilon t} (1 + |p|)^l$$

holds for some real l , depending only on F ;

- 2) in the space S^* there exists a boundary value $F(p_0 + i0, \mathbf{p})$ (respectively $F(p_0 - i0, \mathbf{p})$) of the function $F(p + iq)$ as $q \rightarrow 0$, $q \in \Gamma^+$ (respectively Γ^-).

We shall denote by N the intersection of the classes N^+ and N^- . A function $F(\zeta)$ of class N^+ (respectively N^-) is called an **analytic continuation** of the generalized function $F(p_0 + i0, \mathbf{p})$ (respectively $F(p_0 - i0, \mathbf{p})$).

Let G be an arbitrary open set in R^{n+1} , and let $\tilde{G}$ be a complex neighborhood of G such that, if a ball of radius η belongs to G , then the corresponding complex ball of radius 0.1η belongs to $\tilde{G}$.

Theorem 1. *In order that there exist a function $F(\zeta)$ of class N , holomorphic in the domain $T \cup \tilde{G}$, it is necessary and sufficient that its boundary values $F(p_0 + i0, \mathbf{p})$ and $F(p_0 - i0, \mathbf{p})$: 1) be respectively retarded and advanced functions, and 2) coincide in G .*

The necessity of condition 1) follows from the results of Schwarz ⁽²⁾, Lions ⁽³⁾, and N. N. Bogolyubov and O. S. Parasyuk ⁽⁴⁾ (for $n = 0$). The necessity of condition 2) is obvious.

Sufficiency. From condition 1) it follows ⁽²⁻⁴⁾ that the functions $F(p_0 \pm 0, \mathbf{p})$ admit analytic continuations $F^\pm(\zeta)$ of class $N^\pm$, respectively, in the domains $T^\pm$. From condition 2), by virtue of the “edge of the wedge” theorem*, it follows that the functions $F^\pm(\zeta)$ are holomorphic in $\tilde{G}$ and, consequently, determine a single function $F(\zeta)$, holomorphic in the domain $T \cup \tilde{G}$, of class N .

The function $F(\zeta)$ is the simultaneous analytic continuation of the generalized functions $F(p_0 \pm i0, \mathbf{p})$.

The domain $T \cup \tilde{G}$ in Theorem 1 is not a domain of holomorphy. Therefore the function F is holomorphic in a wider domain—in the envelope of holomorphy $\mathcal{E}(T \cup \tilde{G})$. Hence, in particular, it follows that the set on which the functions $F(p_0 \pm i0, \mathbf{p})$ coincide cannot be arbitrary: it must be a real section of a certain domain of holomorphy. Theorems 2 and 3 are closely connected with this fact.

Theorem 2. *If the commutator $f(p)$ vanishes in the domain G , then it also vanishes in the least convex envelope $B_0(G)$ of the domain G with respect to timelike curves.*

Proof. Construct the advanced and retarded functions**

$$F^+(p) = \int \theta(x_0) \tilde{f}(x) e^{ipx} dx, \quad F^-(p) = - \int \theta(-x_0) \tilde{f}(x) e^{ipx} dx,$$

where $\theta(\xi) = \frac{1}{2}(1 + \text{sign } \xi)$. Then

$$F^+(p) - F^-(p) = (2\pi)^{n+1} f(p).$$

By Theorem 1 there exists a holomorphic function $F(\zeta)$ of class N in $T \cup \widetilde{G}$ such that, in the space S^* , the limiting relations

$$F(p_0 + i\varepsilon, \mathbf{p}) \rightarrow F^+(p), \quad F(p_0 - i\varepsilon, \mathbf{p}) \rightarrow F^-(p) \quad \text{as } \varepsilon \rightarrow +0$$

hold.

It is therefore enough to prove that

$$B_0(G) \subset \text{Re } \mathcal{E}(T \cup \widetilde{G}).$$

Let p_1 and p_2 be arbitrary points of the domain G such that there exists a timelike curve connecting them.

First suppose that this curve lies entirely in G , and let us prove that all timelike curves connecting the points p_1 and p_2 lie entirely in $\mathcal{E}(T \cup \widetilde{G})$. It may happen that all timelike curves connecting p_1 and p_2 lie in G . Allowing for the other possibility, construct a family of timelike curves C_α , depending continuously on the parameter α ($0 \leq \alpha \leq \alpha_0$) and connecting the points p_1 and p_2 , with C_α ($\alpha > 0$) lying entirely inside G , while C_0 contains boundary points of the domain G forming a compact set K . The curves C_α may be assumed analytic, given locally by the equations

$$p_j = p_{j,\alpha}(\xi), \quad j = 0, 1, \dots, n, \quad 0 \leq \alpha \leq \alpha_0, \quad 0 \leq \xi \leq 1.$$

Continue the functions $p_{j,\alpha}(\xi)$ analytically to complex values $\lambda = \xi + i\eta$. By the Heine-Borel lemma, one may assume that the continued functions $p_{j,\alpha}(\lambda)$ are holomorphic in the rectangle

$$\pi_\delta : \quad |\eta| \leq \delta, \quad 0 \leq \xi \leq 1$$

for sufficiently small δ . Thus a family of two-dimensional analytic surfaces F_α has been constructed,

$$\zeta_j = p_{j,\alpha}(\lambda), \quad j = 0, 1, \dots, n, \quad 0 \leq \alpha \leq \alpha_0, \quad \lambda \in \pi_\delta,$$

* This theorem was first proved by N. N. Bogolyubov ⁽⁵⁾, p. 152. For other proofs see ⁽⁶⁻⁸⁾.

** The operation of multiplication by $\theta(x_0)$ of a function of type $\tilde{f}(x)$ can be defined up to a finite linear combination of $\delta^{(k)}(x)$. Therefore the functions $F^\pm(p)$ are defined up to a polynomial.

passing through the curves C_α . For sufficiently small δ , all points ζ on F_α for which $\lambda \in \pi_\delta$ and $\eta \neq 0$ belong to T . Indeed, expanding $p_{j,\alpha}(\lambda)$ in a Taylor series in a neighborhood of the point $\lambda = \xi \in C_\alpha$, we obtain

$$\zeta_j = p_{j,\alpha}(\xi) + p'_{j,\alpha}(\xi)(\lambda - \xi) + O[(\lambda - \xi)^2],$$

whence $\text{Im } \zeta_j = \eta p'_{j,\alpha}(\xi) + O(\eta\delta)$. But the curves C_α are time-like. Therefore $\text{Im } \zeta \in \Gamma$ if $\eta \neq 0$. If, however, $\eta = 0$, then, by construction, $C_\alpha \subset G$ for $\alpha > 0$ and $C_0 \setminus K \subset G$. By the continuity theorem (see ⁽⁹⁾) $K \subset \mathcal{E}(T \cup \widetilde{G})$, and, consequently, the curve C_0 is entirely contained in $\text{Re } \mathcal{E}(T \cup \widetilde{G})$.

Let us consider the other possibility, when there is no time-like curve connecting the points p_1 and p_2 and lying entirely in the domain G . Connect these points by some curve C lying entirely in G , and for all pairs of points on C that can be connected by a time-like curve lying entirely in G , carry out the extension described above. It is not hard to see that, in the end, the original points p_1 and p_2 will also be included among these pairs. The theorem is proved.

Thus it has been proved that $B_0(G) \subset \text{Re } \mathcal{E}(T \cup \widetilde{G})$.

We give three examples of applications of Theorem 2.

- 1) If G is $p^2 < -b^2$, then $B_0(G)$ is $p^2 < 0$; this result was established by another method in ⁽¹⁰⁾.
- 2) If G is $p^2 > -b^2$, then $B_0(G) = R^{n+1}$ and the function $F(\zeta)$ is a polynomial.
- 3) There are no commutators with compact support (except zero).

Theorem 3. *Every function $F(\zeta)$ of class N , holomorphic in the domain $T \cup \widetilde{G}$, where G is an arbitrary domain, is holomorphic in the domain $H(T \cup \widetilde{G})$, consisting of those and only those points ζ for which every complex hyperboloid $(\zeta' - u)^2 = s$ ($u, s \geq 0$ are real parameters), passing through ζ , has with G at least one common interior point.*

For the proof we introduce the function (cf. ^(11, 12))

$$u(p, k) = \frac{1}{(2\pi)^{n+1}} \int \tilde{f}(x) \cos(k\sqrt{x^2}) e^{ipx} dx.$$

Then $u \in S^*$,

$$\left(\frac{\partial^2}{\partial p_0^2} - \frac{\partial^2}{\partial p_1^2} - \dots - \frac{\partial^2}{\partial p_n^2} - \frac{\partial^2}{\partial k^2} \right) u = 0, \quad u \Big|_{k=0} = f(p), \quad \frac{\partial u}{\partial k} \Big|_{k=0} = 0. \quad (1)$$

“Smoothing” $u(p, k)$ with respect to p , we obtain an infinitely differentiable function satisfying the relations (1) with the “smoothed” function $f(p)$. Using Holmgren’s theorem and applying Theorem 2, we conclude that

$$u(p, k) = 0 \quad \text{for} \quad (p, k) \in B_0(G'),$$

where G' is some $(n+2)$ -dimensional neighborhood of the domain G lying in the plane $k=0$. With the aid of (2) we determine the position of the support of the weight function in the integral representation of Jost–Lehmann–Dyson (see ^(8, 11, 12)) for the commutator $F(p_0 + i0, p) - F(p_0 - i0, p)$, and then use known techniques ^(8, 13).

Corollary. If the commutator $f(p)$ vanishes in the domain G , then it also vanishes in the domain $B(G)$ obtained by bending G by the hyperboloids $(p-u)^2 = s$, $s \geq 0$, i.e.

$$B(G) = \text{Re } H(T \cup \widetilde{G}).$$

The domain $H(T \cup \widetilde{G})$ is pseudoconvex.

Remark. Theorem 3 is also valid for such open sets G in which all points of distinct connected components G_j are dis-

separated by space-like intervals, i.e., $(x_i - x_j)^2 < 0$, if $x_i \in G_i$, $x_j \in G_j$, $j \neq i$.

Theorem 3 generalizes previously known results ^(8, 11–13) for domains G bounded by two space-like surfaces to the case of arbitrary domains. Dyson ⁽¹¹⁾ stated a conjecture equivalent to the assertion that Theorem 3 is valid for any open set G .

The inclusion $\mathcal{E}(T \cup \widetilde{G}) \subset H(T \cup \widetilde{G})$ is valid, since $H(T \cup \widetilde{G})$ is the envelope of holomorphy of $T \cup \widetilde{G}$ with respect to holomorphic functions of class N , whereas $\mathcal{E}(T \cup \widetilde{G})$ is the envelope of holomorphy of $T \cup \widetilde{G}$ with respect to all holomorphic functions. It is not clear whether $\mathcal{E}(T \cup \widetilde{G}) = H(T \cup \widetilde{G})$. If, for example, G is the strip $|p_0| < a$, then this is indeed so ^(5, 7),

$$\mathcal{E}(T \cup \widetilde{G}) = H(T \cup \widetilde{G}) : |q| < \left| \text{Im } \sqrt{\xi_0^2 - a^2} \right|,$$

for in this case the domain $T \cup \widetilde{G}$ is semitubular ⁽¹⁴⁾.

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Mathematical Institute named after V. A. Steklov
Academy of Sciences of the USSR

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