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Abstract

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GEOFYSICS

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ON APPROXIMATE SOLUTIONS OF A NON-LINEAR EQUATION FOR THE VELOCITY VORTEX IN THE MIDDLE TROPOSPHERE

(Presented by Academician A. A. Dorodnitsyn on 15 III 1960)

To describe motion in the middle troposphere, one usually uses the equation for the velocity vortex in the geostrophic approximation

$$\Delta \frac{\partial H}{\partial t} + \frac{g}{l}(H, \Delta H + l) = 0, \quad (1)$$

where H is the height of an isobaric surface, g is the acceleration of gravity,

$$l \text{ is the Coriolis parameter, } \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad (A, B) = \frac{\partial A}{\partial x} \frac{\partial B}{\partial y} - \frac{\partial A}{\partial y} \frac{\partial B}{\partial x}.$$

Equation (1) is a test equation for the study of more complicated baroclinic computational schemes.

Considering (1) as an ordinary first-order differential equation in t of the form $y' = f(x, y)$, where in our case $y = \Delta H$, $f(x, y) = -\frac{g}{l}(H, \Delta H + l)$, we shall use for its solution the formula (1)

$$y_{k+1} = y_k + \sum_{\rho=0}^{r-1} A_{r,\rho} f_{r-1-\rho}(x, y), \quad (2)$$

where r is the order of the polynomial, and $A_{r,\rho}$ are the coefficients of this polynomial. We shall consider a polynomial for which $r \leq k+2$; then (2) is rewritten as follows:

$$y_{k+1} = y_k + A_{r,0} f_{k+1}(x, y) + \sum_{\rho=1}^{k+1} A_{k+2,\rho} f_{k+1,\rho}(x, y).$$

Here y_{k+1} is found from the solution of the equation

$$y_{k+1} - A_{r,0} f_{k+1}(x, y_{k+1}) = y_k + \sum_{\rho=1}^{k+1} A_{k+2,\rho} f_{k+1,\rho}(x, y),$$

which in our case takes the form

$$\Delta H_{k+1} + A_{r,0} \frac{g}{l}(H, \Delta H + l)_{k+1} = \Delta H_k + \bar{f}_k(H), \quad (2')$$

where $\bar{f}_k(H)$ is a polynomial of order k in $\frac{g}{l}(H, \Delta H + l)$ at the preceding moments of time. In particular, for the trapezoidal formula $A_{r,0} = \frac{\delta t}{2}$ and

$$\bar{f}_k(H) = -\frac{\delta t g}{2l}(H, \Delta H + l)_k.$$

It should be said that the trapezoidal formula has a somewhat higher accuracy than the improved tangent formula $\left(-\frac{1}{12}\delta t^3 \frac{\partial^3 \Delta H}{\partial t^3}\Big|_{t_1} \text{ and } \frac{1}{3}\delta t^3 \frac{\partial^3 \Delta H}{\partial t^3}\Big|_{t_1} \text{ respectively}\right)$.

Let us introduce dimensionless variables $x = L\bar{x}$, $y = L\bar{y}$ (L is the characteristic scale of variations of H along x, y) and denote $A_{r,0}g\theta/lL^2 = \varepsilon$, $l' = L^2l$, θ being days, n the number of steps in days.

Now equation (2') will take the form:

$$\Delta H_{k+1} + \frac{\varepsilon}{n}(H, \Delta H + l')_{k+1} = F_k. \quad (3)$$

Taking into account the smallness of the parameter ε (of order 10^{-2}), we shall seek the solution of the nonlinear equation (3) in the form of a series in powers of ε

$$H = H_0 + \varepsilon H_1 + \varepsilon^2 H_2 + \dots + \varepsilon^s H_s + \dots \quad (4)$$

Then for H_s we obtain the recurrent system of linear differential equations

$$\begin{aligned} \Delta H_0 &= f_0, \\ \Delta H_1 &= -\frac{1}{n} \left[(H_0, \Delta H_0) + \beta' \frac{\partial H_0}{\partial x} \right], \\ \Delta H_2 &= -\frac{1}{n} \left[(H_0, \Delta H_1) + (H_1, \Delta H_0) + \beta' \frac{\partial H_1}{\partial x} \right], \end{aligned} \quad (5)$$

.....

$$\Delta H_s = -\frac{1}{n} \left[\sum_{i=0}^{s-1} (H_i, \Delta H_{k-1-i}) + \beta' \frac{\partial H_{k-1}}{\partial x} \right].$$

Let us check whether the solution obtained in this way tends to the exact solution. For this purpose we use the solution obtained by E. N. Blinova ⁽³⁾, which for an initial field represented by a superposition of waves of the type $H = -Gy + D \cos(\mu x + \gamma y)$, has the form of the same wave shifted with velocity

$$c = \frac{2\varepsilon}{\theta} \left[G - \frac{\beta'}{\mu^2 + \gamma^2} \right],$$

namely (for one wave)

$$H = -Gy + D \cos(\mu x + \gamma y - ct), \quad t = \frac{\theta}{n}, \quad \gamma = \mu = \frac{2\pi}{L}. \quad (6)$$

For this case equation (3) takes the form

$$\Delta H + \frac{\varepsilon}{n} \left[(H, \Delta H) + \beta' \frac{\partial H}{\partial x} \right] = -(\mu^2 + \gamma^2) D \sqrt{1 - \left(\frac{\mu c \theta}{2n} \right)^2} \cos(\mu x + \gamma y - \varphi).$$

where $\varphi = \arctg(\mu c \theta / 2n)$; then

$$H_0 = -Gy + D \sqrt{1 + \left(\frac{\mu c \theta}{2n} \right)^2} \cos(\mu x + \gamma y - \varphi), \quad (7)$$

$$H_s = \left(\frac{\frac{1}{n} \mu \varepsilon \beta'}{\mu^2 + \gamma^2} \right)^s D \sqrt{1 - \left(\frac{\mu c \theta}{2n} \right)^2} \sin(\mu x + \gamma y - \varphi).$$

Substituting H_0 and H_s into (4), we obtain for the first step

$$H^{(1)} \cong -Gy + D \left\{ \left[1 + \left(\frac{\mu c \theta}{2n} \right)^2 \right] \left[1 + \left(\frac{\frac{1}{n} \mu \varepsilon \beta'}{\gamma^2 + \mu^2} \frac{1 - \left(\frac{1}{n} \mu \varepsilon \beta' / (\mu^2 + \gamma^2) \right)^{s-1}}{1 + \frac{1}{n} \mu \varepsilon \beta' / (\mu^2 + \gamma^2)} \right)^2 \right] \right\}^{1/2} \times$$

$$\times \cos(\mu x + \gamma y - \varphi + \sigma), \quad (8)$$

Figure 1

Figure 1: Figure 1

where

$$\sigma = \arctg \frac{\frac{1}{n}\mu\varepsilon\beta'}{\mu^2 + \gamma^2} \frac{1 - \left(\frac{1}{n}\mu\varepsilon\beta' / (\mu^2 + \gamma^2)\right)^{s-1}}{1 + \frac{1}{n}\mu\varepsilon\beta' / (\mu^2 + \gamma^2)}.$$

In view of the fact that $\frac{1}{n}\mu\varepsilon\beta' / (\mu^2 + \gamma^2) \ll 1$, several approximations are sufficient for obtaining the solution at each step.

After n steps we obtain

$$H^{(n)} \cong -Gy + D \left\{ \left[1 + \left(\frac{\mu c \theta}{2n} \right)^2 \right] \left[1 + \left(\frac{\frac{1}{n}\mu\varepsilon\beta'}{\mu^2 + \gamma^2} \frac{1 - \left(\frac{1}{n}\mu\varepsilon\beta' / (\mu^2 + \gamma^2)\right)^{s-1}}{1 + \frac{1}{n}\mu\varepsilon\beta' / (\mu^2 + \gamma^2)} \right)^2 \right] \right\}^{n/2} \times \\ \times \cos[\mu x + \gamma y - n(\varphi - \sigma)]. \quad (9)$$

It is not difficult to show that the solution (9), as n and $s \rightarrow \infty$, tends to the exact solution (6). For finite n and s , the relative error of the approximate solution is

$$\frac{D^{(n)} - D}{D} \approx \frac{n}{2} \left(\frac{\mu c \theta}{2n} \right)^2 + \frac{n}{2} \left(\frac{\frac{1}{n}\mu\varepsilon\beta'}{\mu^2 + \gamma^2} \right) + O\left(\frac{\varepsilon}{n^2}\right) \approx \frac{n}{2} \left(\frac{\mu c \theta}{2n} \right)^2 + O\left(\frac{\varepsilon}{n}\right);$$

in this case the error due to the inaccuracy of the solution of equation (3) is $< 0.05\%$. Consequently, the principal errors will arise because of the replacement of the time derivative by finite differences. However, the magnitude of the error is half that in work (5), and is equal to $\approx 0.48/n$.

Fig. 1. Initial field

The question of the stability of the finite-difference equation remains unresolved. An approximate estimate can be made for the linearized equation (3), using for this purpose the Courant-Levy method. Representing, in a small neighborhood, the solution in the form $H = \exp[i(\mu x + \gamma y + \nu t)]$, where ν is the frequency (in general a complex quantity), we obtain for the linearized equation the relation

$$\operatorname{tg} \frac{\nu \theta}{2n} = \frac{\theta}{Ln} f_1(H).$$

Figure 2

Figure 2: Figure 2

Since the right-hand side may be any quantity, Ln may also be any quantity, but one restricted by the condition $\varepsilon/n < 1$. Recalling the expression for ε , we obtain $L^2n > 0.3 \cdot 10^{10} \text{ m}^2$. Thus, for example, when $n = 24$, $L > 10 \text{ km}$, and when $n = 1$, $L > 60 \text{ km}$. Taking into account that in meteorological problems L is chosen to be of the order of 250–500 km, we obtain a practically stable solution for any n .

The practical implementation of the considered method of solution is difficult, since it requires retaining all terms of series (4) until the end of the calculation. Generally speaking, it would be more justified to use higher-order formulas, for example $r = 3, 4$, but to solve equation (3) by the method of successive approximations.

Fig. 2. Calculation 24 hours ahead ($n = 8$, $\alpha = 0.25$). The area free from large errors is outlined by a dashed line

Of interest is the usual iterative method, in which H is found from the formula (Liebmann method)

$$H_{ijk+1}^{(\nu+1)} = H_{ijk+1}^{(\nu)} + \alpha [\mathcal{L}_{ijk+1}^{(\nu, \nu+1)} - F_{ijk}], \quad (10)$$

where $\mathcal{L}_{ijk}^{(\nu, \nu+1)}$ is the nonlinear difference operator corresponding to (3), α is the relaxation coefficient, $x = i\delta s$, $y = j\delta s$, ν is the iteration number. For α one can obtain an approximate estimate. Denoting $\delta = H^{(\nu+1)} - H^{(\nu)}$ and taking the difference between formulas (10), written for the $\nu + 1$ and ν iterations, we obtain

$$\delta_{k+1}^{(\nu)} = \delta_{k+1}^{(\nu-1)} + \alpha \left\{ \Delta \delta_{k+1}^{(\nu-1)} + \frac{\varepsilon}{n} [(H, \Delta H + l')_{k+1}^{(\nu)} - (H, \Delta H + l')_{k+1}^{(\nu-1)}] \right\}.$$

According to the Lipschitz theorem, the approximate condition is obtained

$$\begin{aligned} \delta^{(\nu)} &\leq \delta^{(\nu-1)} + \\ &+ \alpha \left\{ \Delta \delta^{(\nu-1)} + \frac{\varepsilon}{n} K \Delta \delta^{(\nu-1)} \right\}, \end{aligned}$$

where

$$K = \left| \frac{(H, \Delta H + l')^{(\nu)} - (H, \Delta H + l')^{(\nu-1)}}{\Delta H^{(\nu)} - \Delta H^{(\nu-1)}} \right|$$

Fig. 3. Calculation 24 hours ahead ($n = 2$, $\alpha = 0.1172$)

Figure 3: Fig. 3. Calculation 24 hours ahead ($n = 2$, $\alpha = 0.1172$)

is the Lipschitz constant.

Fig. 3. Calculation 24 hours ahead ($n = 2$, $\alpha = 0.1172$)

Obviously, the process will converge if

$$0 \leq 4\alpha \left(1 + \frac{\varepsilon}{n}K\right) \leq 1,$$

whence

$$\alpha \leq \frac{0.25}{1 + \frac{\varepsilon}{n}K}.$$

Thus, α depends substantially on the character of the field $H(K)$ and on n .

We carried out computations for a test problem by the trapezoidal formula ($\varepsilon = 0.07$, $A_{r,0} = \theta/2n$). The initial field was taken in the form of a wave (Fig. 1)

$$H_{ij0} = -G'j + D \cos(\mu'i + \gamma'j)$$

($G' = 4$ decameters, $D = 24$ decameters, $\mu' = \gamma' = 2\pi\delta s/L_s$, $L_s = 4000$ km, $\delta s = 250$ km,

for which the exact solution is known⁽³⁾). The exact solution has the form of the same wave shifted to the right by 880 km, which corresponds to $\Delta i = 3.52$. Figures 2 and 3 show the results of computations 24 hours ahead for different n and α . The computations gave a displacement somewhat smaller than $\Delta i = 3.52$. This can be explained by the influence of the artificial boundary conditions, under which the field H on the two outer rows remained constant throughout the entire computation. The retarding influence of the boundaries is clearly visible in the figures presented.

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CITED LITERATURE

1. L. Collatz, *Numerical Methods for the Solution of Differential Equations*, Moscow, 1953.
2. S. S. Tokmalaeva, *Computational Mathematics*, No. 5 (1959).
3. E. N. Blinova, *Applied Mathematics and Mechanics*, 10, issue 5-6 (1946).
4. N. I. Buleev, *Atomic Energy*, 6, issue 3 (1959).
5. S. L. Belousov, *Izv. Academy of Sciences of the USSR, Geophysical Series*, No. 9 (1958).

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