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Abstract

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MATHEMATICS

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ON THE ASYMPTOTICS OF SOLUTIONS OF PROBLEMS WITH RAPIDLY OSCILLATING BOUNDARY CONDITIONS FOR PARTIAL DIFFERENTIAL EQUATIONS

As is known, the solution $u(x, y)$, bounded in the half-plane $y > 0$, of the Laplace equation $\Delta u = 0$ under the boundary condition $u|_{y=0} = \sin kx$ has the form $u = e^{-ky} \sin kx$ and, for large k , differs noticeably from zero only near the boundary.

Phenomena of this kind of boundary effect occur in various problems of mathematical physics with rapidly oscillating boundary conditions. In this article we investigate the asymptotics of solutions of the first boundary-value problem for arbitrary elliptic equations with rapidly oscillating boundary conditions. Some of the possible definitions of rapid oscillation are given in §§ 2 and 3. Since a uniformly bounded sequence of functions oscillating ever more rapidly on the boundary Γ of the domain converges weakly to zero on Γ , and the solution operator for an elliptic equation is continuous, it is easy to derive convergence of the corresponding solutions to zero in every interior subdomain. It is of interest, however, to construct the asymptotics of the solutions near the boundary. We shall construct it by the methods described in the works ^(1,2).

1. Let us first consider, in a plane domain Q with boundary Γ , an elliptic equation of order $2k$

$$L_{2k}u = 0, \quad (1)$$

solvable under arbitrary boundary conditions of the first boundary-value problem. First consider the case when these boundary conditions have the form:

$$\left. \frac{\partial^l u}{\partial n^l} \right|_{\Gamma} = A_l(\varphi) e^{i\omega\varphi} \quad (l = 0, 1, \dots, k-1), \quad (2)$$

where $A_l(\varphi)$ are sufficiently smooth functions that vanish outside some part $\varphi_0 \leq \varphi \leq \varphi_1$ of the boundary (φ is a parameter on the boundary). In some ρ_0 -neighborhood Ω_{φ_0} of this part of the boundary introduce a coordinate system (ρ, φ) (cf. (2)), where ρ , for example, is the distance along the normal to Γ . Passing to these variables, write equation (1) in the form

$$L_{2k}u \equiv H_{2k} \left(\frac{\partial}{\partial \rho}, \frac{\partial}{\partial \varphi} \right) u + L_{2k-1}u = 0, \quad (3)$$

$$H_{2k}(\xi, \eta) = \sum_{j=0}^{2k} a_j(\rho, \varphi) \xi^j \eta^{2k-j}, \quad H_{2k}(i\xi, i\eta) \geq C(\xi^{2k} + \eta^{2k}) \quad (4)$$

for real ξ, η . Expand $a_l(\rho, \varphi)$ in powers of ρ :

$$a_l(\rho, \varphi) = a_l(\varphi) + \sum_{s=1}^p b_{ls}(\varphi) \rho^s + b_{l,p+1}(\rho, \varphi) \rho^{p+1}.$$

Then

$$H_{2k}(\xi, \eta) = M'_0(\xi, \eta) + \dots, \quad M_0(\xi, \eta) = \sum a_j(\varphi) \xi^j \eta^{2k-j}.$$

Accordingly, carrying out analogous expansions of all coefficients of the operator L_{2k} and making the substitution $t = \omega\rho$ and $u = v(\rho, \varphi)e^{i\omega\varphi}$, we represent equation (3) in the form

$$e^{i\omega\varphi} \omega^{2k} \left(M_0 \left(\frac{\partial}{\partial t}, i \right) + \frac{1}{\omega} M_1 + \dots \right) v = 0. \quad (5)$$

Thus, we see that a small parameter $1/\omega$ appears in the equation, which makes it possible to carry out the second iterative process of work (2). Namely, as a first approximation v_0 we take a solution of the ordinary differential equation with respect to t

$$M_0 \left(\frac{\partial}{\partial t}, i \right) v \equiv \sum_{j=0}^{2k} a_j(\varphi) i^{2k-j} \frac{\partial^j v}{\partial t^j} = 0 \quad (6)$$

with coefficients constant with respect to t and characteristic equation

$$M_0(\lambda, i) = \sum a_j(\varphi) i^{2k-j} \lambda^j = 0. \quad (7)$$

From (4) there follows the inequality $M_0(i, i\lambda) > 0$ for all real λ . Therefore, by virtue of Lemma 4 of work (2), equation (7) has exactly k roots

$-\lambda_1, -\lambda_2, \dots, -\lambda_k$ with negative real parts. The general solution of equation (6) of boundary-layer type has the form

$$v_0 = \sum_{l=1}^k c_l(\varphi) \exp(-\lambda_l t) = \sum_{l=1}^k c_l(\varphi) \exp(-\lambda_l \omega \rho); \quad (8)$$

the coefficients $c_l(\varphi)$ are determined uniquely from the condition that v_0 satisfy the boundary conditions (2). The next approximation v_1 ($v = v_0 + \frac{1}{\omega} v_1 + \dots$) is found just as elementarily ⁽²⁾ from the equation $M_0(v_1, i) = -M_1 v_0$ under homogeneous boundary conditions of type (2); analogously, constructing further the second iterative process ⁽²⁾, we obtain successive approximations $v_2, \dots, v_m$. The functions $v_0, v_1, \dots, v_m$ are defined only in a neighborhood of Ω_{ρ_0} . They can be extended to the entire domain ^(1, 2) by multiplying them by a smooth function $\psi(\rho)$; $\psi(\rho) \equiv 1$ for $0 \leq \rho \leq \frac{1}{3}\rho_0$, $\psi(\rho) \equiv 0$ for $\frac{2}{3}\rho_0 < \rho$, and by setting $v_i \equiv 0$ in $Q - \Omega_{\rho_0}$.

Denote

$$z_{m+1} = u - \left(v_0 + \frac{1}{\omega} v_1 + \dots + \frac{1}{\omega^m} v_m \right) e^{i\omega\varphi}, \quad (9)$$

where u is the exact solution of problem (1), (2). We have:

$$|L_{2k} z_{m+1}| < \frac{C}{\omega^{m-2k}}, \quad (10)$$

and z_{m+1} satisfies homogeneous boundary conditions of type (2). Hence, using estimates in the metrics $W_p^{(2k)}$ ⁽³⁾, we obtain

$$\|z_{m+1}\|_{W_p^{(2k)}} \leq \frac{C}{\omega^{m-2k}}. \quad (11)$$

Thus, we obtain the asymptotics of the solution and an estimate of the remainder term, valid for the function u itself and for its derivatives up to order $2k$. With the aid of energy inequalities under smaller restrictions

one can obtain analogous asymptotics and estimates up to derivatives of order $\leq k$.

2. Let a family of functions $\{f_\varepsilon\}$, depending on the parameter ε , be given on Γ . We shall say that this family is $\frac{1}{\varepsilon}$ -oscillating on the interval $\mu(\varphi_0 \leq \varphi \leq \varphi_1)$, $\mu \subset \Gamma$, if for any φ from this interval

$$\left| \int_{\varphi_0}^{\varphi} f_\varepsilon(\varphi) d\varphi \right| < K\varepsilon \quad (K \text{ does not depend on } \varepsilon). \quad (12)$$

The family $\{f_\varepsilon\}$ $\frac{1}{\varepsilon}$ -oscillates on the whole boundary Γ , if Γ can be covered by a finite number of intervals μ_i ($i = 1, \dots, p$), in each of which it $\frac{1}{\varepsilon}$ -oscillates. With the aid of a partition of unity $1 \equiv \sum \psi_i$, where $\psi_i \equiv 0$ everywhere outside μ_i , the function v_ε is represented in the form $v_\varepsilon \equiv \sum \psi_i v_{i\varepsilon}$, where each term is different from 0 only in μ_i .

We shall seek the asymptotics of the solutions u_ε of equation (1) under the boundary conditions

$$\left. \frac{\partial^l u_\varepsilon}{\partial \rho^l} \right|_\Gamma = \alpha_l(\varphi) g_{l\varepsilon}(\varphi), \quad (13)$$

where $\alpha_l(\varphi) \equiv 0$ outside one of the intervals μ_i , and $g_{l\varepsilon}(\varphi)$ are $\frac{1}{\varepsilon}$ -oscillating functions on μ_i . The general case of oscillating boundary conditions is obtained by summing such solutions over all intervals μ_i . We shall regard μ_i as corresponding to the interval $(0, 2\pi)$ of the variable φ .

Expand the function into a Fourier series

$$g_{l\varepsilon} = \sum_{k=-\infty}^{+\infty} a_{lk}(\varepsilon) e^{ik\varphi}$$

and solve, for each term, the boundary-value problem (1), (2) with $A_l(\varphi) = \alpha_l(\varphi) a_{lk}(\varepsilon)$, $\omega = k$. In this way we obtain successive approximations $v_{0k} e^{ik\varphi}, \dots, \frac{1}{k^m} v_{mk} e^{ik\varphi}$, and, summing them, obtain the asymptotics of u_ε in the form

$$w_0 + w_1 + \dots + w_m, \quad \text{where } w_s = \sum_{k=-\infty}^{+\infty} \frac{1}{k^s} v_{sk} e^{ik\varphi}. \quad (14)$$

\end{equation}

From condition (12) of rapid oscillation of the function $g_{l\varepsilon}(\varphi)$ it follows that

$$|a_{lk}(\varepsilon)| \leq C|k|\varepsilon.$$

Therefore the terms v_{sk} corresponding to small values of k are small, while the terms with large values of k decay rapidly as ρ increases, by virtue of (8).

The following estimates hold:

$$|w_0| \leq C \frac{\varepsilon}{\rho} \quad \text{for } \rho > \varepsilon, \quad |w_1| \leq C \min(\varepsilon^{1/2}, \varepsilon |\ln \rho|),$$

$$|w_k| \leq C\varepsilon \quad \text{for } k \geq 2, \quad |D^s w_{s+1}| \leq C \min(\varepsilon^{1/3}, \varepsilon |\ln \rho|), \quad (15)$$

$$|D^s w_p| \leq C\varepsilon \quad \text{for } p > s + 1.$$

It can be verified that the remainder $z_\varepsilon = u_\varepsilon - (w_0 + \dots + w_{2k})$ satisfies an equation of the form

$$L_{2k} z_\varepsilon = O(\varepsilon)$$

and homogeneous boundary conditions. Therefore, by (3),

$$\|z_\varepsilon\|_{W_p^{(2k)}} \leq C\varepsilon.$$

Thus, the iterations given above yield a remainder that is small throughout the whole domain together with its derivatives up to order $2k$.

If we want to obtain a remainder that is small in the metric L_p ($p > 1$), then, as is clear from (15), it is enough to restrict ourselves to the first approximation w_0 : $u_\varepsilon = w_0 + z_{\varepsilon 0}$ ($\|z_{\varepsilon 0}\|_{L_p} \leq C\varepsilon$). If we want to ensure that z_ε is small of order ε in the metric C , then, as is clear from (14), it is enough to restrict ourselves to the first two terms $w_0 + w_1$.

3. One can give a more refined definition of oscillation on the interval $a \leq \varphi \leq b$. A family of functions $\{f_\varepsilon\}$ on the interval $[a, b]$ is called $\frac{1}{\varepsilon}$ -oscillating of order s if:

1)

$$\left| \int_{a_\varepsilon}^{b_\varepsilon} f_\varepsilon d\varphi \right| \leq C\varepsilon^s, \quad \text{where } |a - a_\varepsilon| \leq C\varepsilon, \quad |b - b_\varepsilon| \leq C\varepsilon;$$

- 2) for the s -th successive principal primitive* the condition

$$\left| \int_{a_\varepsilon}^\varphi f_\varepsilon^{(-s)} d\varphi \right| \leq C\varepsilon^s$$

is satisfied.

If the family of boundary conditions $g_{1\varepsilon}(\varphi)$ is $\frac{1}{\varepsilon}$ -oscillating of order s , then, applying the preceding constructions, we obtain the estimates: for $l < s$, $|w_l| \leq C\varepsilon^s / \rho^{s-l}$ ($\rho > \varepsilon$), $|w_s| \leq C \min(\varepsilon^s, \varepsilon |\ln \rho|)$, $|w_m| \leq C_m \varepsilon^s$ for $m > s$, and the asymptotic representation (14) can be carried to remainder terms of order ε^s .

4. The multidimensional case does not differ in any way from the two-dimensional one. Let us note that it is enough to have rapid oscillations in at least one direction on the boundary for the constructions and estimates given above to be valid. For example, the definition of $\frac{1}{\varepsilon}$ -oscillation given in item 2 is generalized as follows: a family of functions $\{f_\varepsilon\}$ is called $\frac{1}{\varepsilon}$ -oscillating on a given part of the boundary if, in some local coordinate system, the integrals over any parallelepipedal domain are of order ε . As in the two-dimensional case, we pass from a local definition to a definition of oscillation over the entire boundary.

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* We shall call the primitive $f_\varepsilon^{(-1)}$ of the function f_ε the principal primitive for which

$$\int_{a_\varepsilon}^{b_\varepsilon} f_\varepsilon^{(-1)} d\varphi = 0.$$

Note: Figure translations are in progress. See original paper for figures.

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