

# On the impossibility of constructing a linear polynomial operator furnishing an approximation within the order of the best approximation

D. L. Berman

1958

SovietRxiv

---

View the original and related papers at <https://sovietrxiv.org/items/ru-195801.73813>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

**Abstract**

**Full Text**

**D. L. BERMAN**

**ON THE IMPOSSIBILITY OF CONSTRUCTING A  
LINEAR POLYNOMIAL OPERATOR GIVING AN  
APPROXIMATION OF THE ORDER OF THE BEST  
APPROXIMATION**

*(Presented by Academician A. N. Kolmogorov on 13 II 1958)*

**Mathematics**

1°. For definiteness we shall first consider the space  $\widetilde{C}$  of all continuous  $2\pi$ -periodic functions  $f(x)$  with norm

$$\|f\|_{\widetilde{C}} = \max_{0 \leq x < 2\pi} |f(x)|.$$

Put  $f_t(x) = f(x + t)$ .

It is known that for every  $f \in \widetilde{C}$  there exists a unique polynomial of best approximation of order  $n$ , which we shall denote by  $r_n(f, x)$ . The polynomial  $r_n(f, x)$  may be regarded as an operator that assigns to each  $f \in \widetilde{C}$  its polynomial of best approximation of order  $n$ . The operator  $r_n(f, x)$  is nonlinear.

At present there is as yet no method for computing polynomials of best approximation; therefore the following question is natural:

Does there exist a **linear operator**  $U_n(f, x)$  possessing the properties:

- 1)  $U_n(f, x)$  maps  $\widetilde{C}$  into  $\widetilde{C}$ ;
- 2) for every  $f \in \widetilde{C}$ ,  $U_n(f, x)$  is a polynomial of order  $\leq n$ ;
- 3) for every  $f \in \widetilde{C}$ ,

$$\|f(x) - U_n(f, x)\| = O(E_n),$$

where  $E_n(f) = E_n = \|f(x) - r_n(f, x)\|$ ?

In the present note it is proved that this question must be answered in the negative.

2°. **Theorem 1.** *There does not exist a linear operation  $U_n(f, x)$  satisfying conditions 1)–3).*

**Proof.** Suppose that there exists an operation  $U_n(f, x)$  satisfying conditions 1)–3). Then from condition 3), taking into account that  $E_n(f) = 0$  when  $f$  is a trigonometric polynomial of order not exceeding  $n$ , it follows that

$$U_n(f, x) \equiv f(x), \quad (1)$$

when  $f$  is a trigonometric polynomial of order not exceeding  $n$ .

But it is known <sup>(1)</sup> that for every linear operation satisfying conditions 1)–2) and equality (1), the equality

$$\frac{1}{2\pi} \int_0^{2\pi} U_n(f_t, x - t) dt = S_n(f, x), \quad (2)$$

holds, where  $S_n(f, x)$  is the partial sum of order  $n$  of the Fourier series of the function  $f(x)$ .

It is easy to see that

$$\frac{1}{2\pi} \int_0^{2\pi} (f_t, x - t) dt = f(x), \quad (3)$$

where  $(f, x)$  is the value of the function  $f$  at the point  $x$ .

From (2) and (3) it follows that

$$\frac{1}{2\pi} \int_0^{2\pi} (U_n(f_t) - f_t, x - t) dt = S_n(f, x) - f(x). \quad (4)$$

Since the operator  $U_n(f, x)$  satisfies condition 3), there exists a constant  $C$  such that

$$\|U_n(f_t) - f_t\| \leq CE_n(f_t), \quad -\infty < t < \infty. \quad (5)$$

We note that

$$E_n(f_t) = E_n(f), \quad -\infty < t < \infty.$$

Therefore inequality (5) takes the form

$$\|U_n(f_t, x) - f_t(x)\| \leq CE_n(f), \quad -\infty < t < \infty,$$

and then from equality (4) it follows that

$$|S_n(f, x) - f(x)| \leq CE_n(f), \quad -\infty < x < \infty. \quad (6)$$

Inequality (6) contradicts the classical result of du Bois-Reymond from the theory of Fourier series. Theorem 1 is proved.

An analogous theorem can be proved in the case of the space  $C$  of all functions  $f(x)$  continuous on the segment  $[-1, 1]$ , with norm  $\|f\| = \max_{-1 \leq x \leq 1} |f(x)|$ .

3°. In (2) it is proved that formula (2) is valid for functional spaces of type  $E$ , a particular case of which is the space  $\widetilde{C}$ . Therefore Theorem 1 admits the following generalization:

**Theorem 2.** *Let a functional space of type  $E$  be such that, for at least one function  $f \in E$ , the relation*

$$\overline{\lim}_{n \rightarrow \infty} \|f - S_n(f)\| = \infty$$

*holds. Then there does not exist a linear operation  $U_n(f, x)$  satisfying the conditions:*

- 1)  $U_n(f, x)$  maps  $E$  into  $E$ ;
- 2) for every  $f \in E$ ,  $U_n(f, x)$  is a polynomial of order  $\leq n$ ;
- 3) for every  $f \in E$ ,

$$\|U_n(f) - f\|_E = O(E_n(f)),$$

where

$$E_n(f) = \inf_{T_n \in \Pi_n} \|f - T_n\|_E$$

and  $T_n$  ranges over the set  $\Pi_n$  of all trigonometric polynomials of order  $\leq n$ .

The space  $\widetilde{L}$  of all summable  $2\pi$ -periodic functions with norm

$$\|f\| = \frac{1}{2\pi} \int_0^{2\pi} |f(t)| dt$$

is a functional space of type  $E$ . Moreover, it is known that there exist functions  $f \in \widetilde{L}$  satisfying the condition

$$\overline{\lim}_{n \rightarrow \infty} \int_0^{2\pi} |f - S_n(f)| dx = \infty.$$

Therefore, from Theorem 2 it follows:

**Theorem 3.** There does not exist a linear operation satisfying the conditions:

- 1)  $U_n(f, x)$  maps  $\widetilde{L}$  into  $\widetilde{L}$ ;
- 2) for every  $f \in \widetilde{L}$ ,  $U_n(f, x)$  is a trigonometric polynomial of order  $\leq n$ ;
- 3) for every  $f \in \widetilde{L}$ ,

$$\int_0^{2\pi} |f(x) - U_n(f, x)| dx = O(E_n(f)),$$

where

$$E_n = E_n(f) = \inf_{T_n \in \Pi_n} \frac{1}{2\pi} \int_0^{2\pi} |f(x) - T_n(x)| dx.$$

**Remark.** It is known <sup>(3)</sup> that formula (2) generalizes to the case of functions of many variables. In view of this, Theorem 2 can also be extended to the case of functions of many variables.

4°. It is well known that, for any fixed  $p$  satisfying the inequalities  $0 < p < 1$ , one can construct a linear polynomial operator having properties 1) and 2) and satisfying, for every  $f \in \widetilde{C}$ , the condition

$$\|f(x) - U_n(f, x)\| = O(E_{[pn]}(f)),$$

where  $[pn]$  is the integer part of the number  $pn$  <sup>(4, 5)</sup>.

Therefore, in connection with Theorem 1 the following problem arises.

Let an arbitrary sequence of positive numbers  $\{p_n\}_{n=1}^{\infty}$  be given, satisfying the inequalities  $0 < p_n \leq 1$ ,  $n = 1, 2, \dots$ , and let  $\lim_{n \rightarrow \infty} p_n = 1$ . The question is whether one can construct a linear polynomial operator  $U_n(f, x)$  having properties 1) and 2) and satisfying, for every  $f \in \widetilde{C}$ , the condition

$$3') \quad \|f(x) - U_n(f, x)\| = O(E_{[p_n n]}(f)).$$

The solution of this problem is given by the theorem:

**Theorem 4.** There does not exist a linear operation  $U_n(f, x)$  satisfying conditions 1)–2) and 3').

In connection with Theorem 3 the following theorem is appropriate:

**Theorem 5.** For any fixed  $f \in \widetilde{C}$  one can construct such a linear polynomial operator  $U_n(f, x)$  of order  $n$  that

$$|f(x) - U_n(f, x)| \leq E_n(f), \quad n = 0, 1, 2, \dots, \quad -\infty < x < \infty. \quad (7)$$

This theorem follows directly from Marcinkiewicz' s theorem <sup>(6)</sup>, according to which, for any  $f \in \widetilde{C}$ , one can find such a matrix of interpolation nodes that the corresponding Lagrange interpolation process satisfies inequality (7).

Novgorod State  
Pedagogical Institute

Received  
2 I 1958

## CITED LITERATURE

- <sup>1</sup> D. L. Berman, DAN, **85**, No. 1 (1952).
- <sup>2</sup> D. L. Berman, DAN, **88**, No. 1 (1953).
- <sup>3</sup> D. L. Berman, DAN, **91**, No. 6 (1953).
- <sup>4</sup> S. N. Bernstein, Izv. AN SSSR, OMEN, No. 9, 1151 (1931).
- <sup>5</sup> D. L. Berman, DAN, **109**, No. 4 (1956).
- <sup>6</sup> I. P. Natanson, *Constructive Theory of Functions*, 1949.

*Note: Figure translations are in progress. See original paper for figures.*

*Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.*