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Abstract

Full Text

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A SUFFICIENT CONDITION FOR THE CONVERGENCE OF THE METHOD OF AVERAGING FUNCTIONAL CORRECTIONS

(Presented by Academician N. N. Bogolyubov, 13 V 1958)

MATHEMATICS

Consider the linear Fredholm integral equation of the second kind

$$y(x) = \varphi(x) + \lambda \int_a^b K(x, \xi) y(\xi) d\xi \quad (0 < |\lambda| < \infty). \quad (1)$$

Assume that the functions $\varphi(x)$ and $K(x, \xi)$ are real and belong to the space $L^2(a, b)$. Let equation (1) have a unique solution for a certain value of the parameter λ .

The method of averaging functional corrections, set forth in ⁽¹⁻⁴⁾, consists in the following: in the first approximation we put

$$y_1(x) = \varphi(x) + \lambda \alpha_1 \int_a^b K(x, \xi) d\xi, \quad (2)$$

where

$$\alpha_1 = \frac{1}{h} \int_a^b y_1(x) dx \quad (h = b - a > 0). \quad (3)$$

From equalities (2) and (3) we determine α_1 :

$$\alpha_1 = \frac{1}{D(\lambda)} \int_a^b \varphi(x) dx, \quad (4)$$

where

$$D(\lambda) = h - \lambda \int_a^b \int_a^b K(x, \xi) d\xi dx.$$

In the n -th approximation we put

$$y_n(x) = \varphi(x) + \lambda \int_a^b K(x, \xi)(y_{n-1}(\xi) + \alpha_n) d\xi, \quad (5)$$

where

$$\alpha_n = \frac{1}{h} \int_a^b \delta_n(x) dx; \quad (6)$$

$$\delta_n(x) = y_n(x) - y_{n-1}(x) \quad (n = 2, 3, \dots). \quad (7)$$

From equalities (5), (6), and (7) we obtain

$$\delta_n(x) = \lambda \int_a^b K(x, \xi)(\delta_{n-1}(\xi) - \alpha_{n-1}) d\xi + \lambda \alpha_n \int_a^b K(x, \xi) d\xi; \quad (8)$$

$$\alpha_n = \frac{\lambda}{D(\lambda)} \int_a^b \int_a^b K(x, \xi)(\delta_{n-1}(\xi) - \alpha_{n-1}) d\xi dx \quad (n = 2, 3, \dots). \quad (9)$$

Let $D(\lambda) \neq 0$. Then from the assumptions concerning the functions $\varphi(x)$ and $K(x, \xi)$ it follows that all the functions $y_n(x)$, and hence also all the functions $\delta_n(x)$, belong to the space $L^2(a, b)$.

The functions $\delta_n(x)$ and α_n can be represented in the form

$$\delta_n(x) = \lambda \int_a^b (K(x, \xi) - M(x))(\delta_{n-1}(\xi) - \lambda h \alpha_{n-1} t) d\xi + \lambda \alpha_n h M(x); \quad (8')$$

$$\alpha_n = \frac{\lambda}{D(\lambda)} \int_a^b \int_a^b (K(x, \xi) - M(x))(\delta_{n-1}(\xi) - \lambda h \alpha_{n-1} t) d\xi dx, \quad (9')$$

where

$$M(x) = \frac{1}{h} \int_a^b K(x, \xi) d\xi,$$

t is an arbitrary parameter.

Subtracting $\lambda h \alpha_n t$ from both sides of equality (8'), squaring the result obtained, and integrating with respect to x , we then obtain

$$\Phi_n(t) = \int_a^b (\delta_n(x) - \lambda h \alpha_n t)^2 dx =$$

$$= \lambda^2 \int_a^b \left\{ \int_a^b (K(x, \xi) - M(x))(\delta_{n-1}(\xi) - \lambda h \alpha_{n-1} t) d\xi + \alpha_n h (M(x) - t) \right\}^2 dx.$$

Using Minkowski's inequality, we obtain:

$$\begin{aligned} \{\Phi_n(t)\}^{1/2} &\leq |\lambda| \left\{ \int_a^b \left[\int_a^b (K(x, \xi) - M(x))(\delta_{n-1}(\xi) - \lambda h \alpha_{n-1} t) d\xi \right]^2 dx \right\}^{1/2} + \\ &\quad + |\lambda| h \left\{ \alpha_n^2 \int_a^b (M(x) - t)^2 dx \right\}^{1/2}. \end{aligned}$$

Applying the Cauchy-Bunyakovsky inequality, we finally obtain the inequality

$$\begin{aligned} \{\Phi_n(t)\}^{1/2} &\leq |\lambda| \left\{ \int_a^b \int_a^b (K(x, \xi) - M(x))^2 d\xi dx \right\}^{1/2} \cdot \{\Phi_{n-1}(t)\}^{1/2} + \\ &\quad + |\lambda| h \left\{ \alpha_n^2 \int_a^b (M(x) - t)^2 dx \right\}^{1/2}. \end{aligned} \quad (10)$$

From (9') we have:

$$\alpha_n^2 \leq \frac{\lambda^2 h}{D^2(\lambda)} \Phi_{n-1}(t) \int_a^b \int_a^b (K(x, \xi) - M(x))^2 d\xi dx. \quad (11)$$

On the basis of (11), from (10) we obtain the relation

$$\begin{aligned} \{\Phi_n(t)\}^{1/2} &\leq |\lambda| \{\Phi_{n-1}(t)\}^{1/2} \left\{ \int_a^b \int_a^b (K(x, \xi) - M(x))^2 d\xi dx \right\}^{1/2} \times \\ &\quad \times \left\{ 1 + \frac{h^{3/2} |\lambda|}{|D(\lambda)|} \left[\int_a^b (M(x) - t)^2 dx \right]^{1/2} \right\}. \end{aligned}$$

Put

$$t = K = \frac{1}{h^2} \int_a^b \int_a^b K(x, \xi) d\xi dx;$$

then

$$\Phi_n(K) \leq \mathcal{L}^2 \Phi_{n-1}(K), \quad (12)$$

where

$$\mathcal{L}^2 = \lambda^2 \int_a^b \int_a^b (K(x, \xi) - M(x))^2 d\xi dx \left\{ 1 + \frac{h^{3/2} |\lambda|}{|D(\lambda)|} \left[\int_a^b (M(x) - K)^2 dx \right]^{1/2} \right\}^2,$$

or

$$\mathcal{L}^2 = \lambda^2 (B^2 - hM^2) \left\{ 1 + \frac{h^{3/2} |\lambda|}{|D(\lambda)|} [M^2 - hK^2]^{1/2} \right\}^2;$$

$$B^2 = \int_a^b \int_a^b K^2(x, \xi) d\xi dx; \quad (13)$$

$$M^2 = \frac{1}{h^2} \int_a^b \left(\int_a^b K(x, \xi) d\xi \right)^2 dx.$$

Since the functions $\varphi(x)$, $K(x, \xi)$ belong to the space $L^2(a, b)$, it follows from equalities (2) and (4) ($D(\lambda) \neq 0$) that

$$\Phi_1(K) \leq C \quad (\delta_1(x) = y_1(x)). \quad (14)$$

Let $\mathcal{L}^2 < 1$; then from (12) and (14) it follows that, as $n \rightarrow \infty$, $\Phi_n(K) \rightarrow 0$; consequently, by (11), $\alpha_n^2 \rightarrow 0$, and hence also $\alpha_n \rightarrow 0$.

From

$$\Phi_n(K) = \int_a^b (\delta_n(x) - \lambda h K \alpha_n)^2 dx$$

it follows that, as $n \rightarrow \infty$,

$$\int_a^b \delta_n^2(x) dx \rightarrow 0;$$

i.e. the sequence of functions $y_n(x)$ converges in itself. Since the space $L^2(a, b)$ is complete, it follows that the sequence of functions $y_n(x)$ converges, as $n \rightarrow \infty$, to a function $Y(x)$ belonging to the space $L^2(a, b)$. It is evident that the function $Y(x)$ is a solution of equation (1).

The derived condition $\mathcal{L}^2 < 1$ is less restrictive than the condition given in paper (2).

In the cases $K(x, \xi) \equiv C$ and $K(x, \xi) \equiv K(x)$, $B^2 - hM^2 = 0$, $\mathcal{L}^2 = 0$, $\alpha_2 = 0$; the first approximation gives the exact solution.

If

$$\int_a^b K(x, \xi) d\xi = 0,$$

then $\mathcal{L}^2 = \lambda^2 B^2$. In this case the method of averaging functional corrections degenerates into the method of successive approximations. For $\lambda^2 B^2 < 1$, as is known, the convergence of the method of successive approximations has been proved.

Example. Consider the simple equation

$$y(x) = -20.2\sqrt{x} + 3 \int_0^1 \sqrt{x} (\xi + 10)y(\xi) d\xi,$$

which has the obvious solution $y(x) = \sqrt{x}$.

For this example we have: $B^2 = \frac{331}{6}$; $M^2 = \frac{441}{8}$; $K^2 = 49$; $D(\lambda) = 1 -$

$$-3 \cdot 7 = -20; \quad \mathcal{L}^2 = \frac{3}{8} \left(1 + \frac{21}{80} \sqrt{2} \right)^2 < 1.$$

It should be noted that the usually employed sufficient condition for convergence of the method of successive approximations is not satisfied in the present case, since $\lambda^2 B^2 = \frac{331}{6} \cdot 9 > 1$. The ordinary process of successive approximations diverges in this example. However, by the method of averaging functional corrections this equation is solved. The first and second approximations have the form

$$y_1(x) = \sqrt{x} + 0.01\sqrt{x};$$

$$y_2(x) = \sqrt{x} - 0.0001\sqrt{x}.$$

$$y_1(x) - y_2(x) = 0.0001\sqrt{x},$$

i.e., the relative error is 0.01%.

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