

Estimates for Solutions of Elliptic and Parabolic Systems

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1958

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Abstract

Full Text

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ESTIMATES OF SOLUTIONS OF ELLIPTIC AND PARABOLIC SYSTEMS

(Presented by Academician V. I. Smirnov, 18 I 1958)

MATHEMATICS

1. In the article ⁽¹⁾ the spaces $W_{x(1), \dots, x(r), p}^{(l_1, \dots, l_r)}(Q)$ were introduced for $p = 2$ and arbitrary nonnegative $l_1, \dots, l_r$. Keeping the notation adopted there, we introduce the indicated spaces for arbitrary p ($1 < p < +\infty$).

Let first l_k be a nonnegative integer. We shall say that

$f = f(x) \in W_{x(k), p}^{(l_k)}(Q)$ if f has generalized derivatives with respect to $x^{(k)}$, summable over Q to the power p , up to order l_k . In this case we set:

$$\|f\|_{W_{x(k), p}^{(l_k)}(Q)} = \left[\sum_{q \leq l_k} \int_Q |D_{x(k)}^q f|^p dx \right]^{1/p}.$$

Let now $l_k = l'_k + \lambda_k$. We shall say that $f \in W_{x(k), p}^{(l_k)}(Q)$ if

$f \in W_{x(k), p}^{(l'_k)}(Q)$ and if for all $q \leq l'_k$

$$L_{k, p}(D_{x(k)}^q f) = \int_{Q \times \Omega^{(k)}} |\Delta(x^{(k)}, y^{(k)}) D_{x(k)}^q f|^p \frac{dx dy^{(k)}}{|x^{(k)} - y^{(k)}|^{n_k + p\lambda_k}} < +\infty.$$

In this case

$$\|f\|_{W_{x(k), p}^{(l_k)}(Q)} = \left\{ \|f\|_{W_{x(k), p}^{(l'_k)}(Q)}^p + \sum_{q \leq l'_k} L_{k, p}(D_{x(k)}^q f) \right\}^{1/p}.$$

We shall further say that

$f \in W_{x(1), \dots, x(r), p}^{(l_1, \dots, l_r)}(Q)$ if

$f \in W_{x(k), p}^{(l_k)}(Q)$ for all $k = 1, 2, \dots, r$. In this case

$$\|f\|_{W_{x(1), \dots, x(r), p}^{(l_1, \dots, l_r)}(Q)} = \left\{ \sum_{k=1}^r \|f\|_{W_{x(k), p}^{(l_k)}(Q)}^p \right\}^{1/p}.$$

For $r = 1$ and $Q = \Omega \subset E_n$, the space $W_{x,p}^{(l)}(\Omega)$ will be denoted simply by $W_p^{(l)}(\Omega)$. Likewise, when $l_1 = l_2 = \dots = l_r = l$, we shall denote $W_{x(1),\dots,x(r),p}^{(l,\dots,l)}(Q)$ by $W_p^{(l)}(Q)$.

Theorem 1. *Let the domain Q be bounded by sufficiently smooth surfaces and let*

$f \in W_{x(1),\dots,x(r),p}^{(l_1,\dots,l_r)}(Q)$. Then for any integral and nonnegative $m_1, \dots, m_r$, satisfying the inequality

$$\mu = 1 - \sum_{k=1}^r \frac{m_k}{l_k} \geq 0,$$

there exist generalized derivatives $D_{x(1)}^{m_1} \dots D_{x(r)}^{m_r} f$, belonging to $W_{x(1),\dots,x(r),p}^{(\bar{l}_1,\dots,\bar{l}_r)}(Q)$ with $\bar{l}_k = l_k \mu$ ($k = 1, 2, \dots, r$). Moreover

$$\|D_{x(1)}^{m_1} \dots D_{x(r)}^{m_r} f\|_{W_{x(1),\dots,x(r),p}^{(\bar{l}_1,\dots,\bar{l}_r)}(Q)} \leq C \|f\|_{W_{x(1),\dots,x(r),p}^{(l_1,\dots,l_r)}(Q)} \quad (1)$$

with C depending only on Q .

It follows from this theorem that, for integer l , the space $W_p^{(l)}(Q)$ coincides with the corresponding space of S. L. Sobolev.

It would be interesting to obtain, for arbitrary p , embedding theorems analogous to those given in (1) for $p = 2$. Unfortunately, this has not yet been accomplished.

2. Let Ω be a finite or infinite domain of n -dimensional space E_n , whose boundary is a sufficiently smooth $(n-1)$ -dimensional finite or infinite surface without edge S . Consider in Ω the linear differential operator

$$L\left(x, \frac{\partial}{\partial x}\right)u = \sum_{r=1}^{2k} \sum_{i_1,\dots,i_r=1}^n A^{(i_1,\dots,i_r)}(x) \frac{\partial^r u}{\partial x_{i_1} \dots \partial x_{i_r}} + A(x)u,$$

where $A^{(i_1,\dots,i_r)}(x)$ and $A(x)$ are square matrices of order N , and $u = u(x)$ is a vector function with N components. Denote by $L_0(x, \partial/\partial x)$ the principal part of $L(x, \partial/\partial x)$, and by $L_0(x, i\alpha)$ the matrix obtained from $L_0(x, \partial/\partial x)$ by replacing the symbols $\partial/\partial x_s$ by the expressions $i\alpha_s$ ($s = 1, 2, \dots, n$). Following I. G. Petrovskii (2), we shall say that $L(x, \partial/\partial x)$ is elliptic in Ω if, for every $x \in \Omega$ and any real $\alpha_1, \dots, \alpha_n$, the determinant $|L_0(x, i\alpha)|$ of the matrix $L_0(x, i\alpha)$ satisfies the inequality:

$$|L_0(x, i\alpha)| \geq \delta |\alpha|^{2k} \quad (\delta > 0, \quad |\alpha| = \sqrt{\alpha_1^2 + \dots + \alpha_n^2}).$$

Suppose that S can be covered by a finite number of overlapping surfaces σ_s ($s = 1, 2, \dots, q$) such that each σ_s can be specified by a sufficiently regular parametric equation $x = x(\gamma')$ ($\gamma' = (\gamma_1, \dots, \gamma_{n-1})$). Denote by $\nu = \nu(\gamma')$ the unit normal vector to S at the point $x' = x(\gamma')$, and assume that

$$x = x(\gamma') + \nu(\gamma')\gamma_n \quad (2)$$

establishes a one-to-one and sufficiently regular correspondence between some domain of the space of points $\gamma = (\gamma_1, \dots, \gamma_n)$ and some n -dimensional neighborhood of the surface σ_s . Specify on S k linear differential operators $R_\mu(x', \partial/\partial x)$ of orders m_μ ($\mu = 1, 2, \dots, k$). Using (2), transform $L(x, \partial/\partial x)$ and $R_\mu(x', \partial/\partial x)$ to the new variables $\gamma_1, \dots, \gamma_n$. Denote the results of this transformation by $L(\gamma, \partial/\partial \gamma)$ and $R_\mu(\gamma, \partial/\partial \gamma)$, and their principal parts by $L_0(\gamma, \partial/\partial \gamma)$ and $R_\mu^{(0)}(\gamma', \partial/\partial \gamma)$ ($\mu = 1, 2, \dots, k$).

Denote by $D(\gamma'; \beta')$ ($\beta' = (\beta_1, \dots, \beta_{n-1})$) the matrix of order kN :

$$D(\gamma'; \beta') = \left\| \int_{C(\beta')} R_\mu^{(0)}(\gamma', i\beta) L_0^{-1}(\gamma', i\beta) \beta_n^{\lambda-1} d\beta_n \right\|_{\lambda, \mu=1, 2, \dots, k} \quad (\beta = (\beta_1, \dots, \beta_n)),$$

where $C(\beta')$ is a positively oriented contour, situated in the upper half-plane of the complex β_n -plane and enclosing all roots lying there of the equation $|L_0(\gamma', i\beta)| = 0$ for fixed γ' and β' .

We shall say that L and R_μ satisfy on S condition (L) (of B. Ya. Lopatinskii (3)) if, for every $x' \in S$ and any real $\beta_1, \dots, \beta_{n-1}$, for which

$$|\beta'| = \sqrt{\beta_1^2 + \dots + \beta_{n-1}^2} = 1,$$

the inequality

$$\|D(\gamma', \beta')\| \geq \delta' \quad (\delta' > 0)$$

holds.

Theorem 2. Suppose: 1) L is elliptic in Ω , and L and R_μ ($\mu = 1, 2, \dots, k$) satisfy condition (L) on S ; 2) $l \geq 2k$, $m_\mu < l - \frac{1}{2}$ ($\mu = 1, 2, \dots, k$). Then, for every vector-function $u \in W_2^{(l)}(\Omega)$, the inequality

$$C_1 \left[\|Lu\|_{W_2^{(l-2k)}(\Omega)} + \sum_{\mu=1}^k \|R_\mu u\|_{W_2^{(l-m_\mu-\frac{1}{2})}(S)} \right] \leq \|u\|_{W_2^{(l)}(\Omega)} \leq C_2 \left[\|Lu\|_{W_2^{(l-2k)}(\Omega)} + \sum_{\mu=1}^k \|R_\mu u\|_{W_2^{(l-m_\mu-\frac{1}{2})}(S)} + \|u\|_{L_2(\Omega)} \right]$$

where C_1 and C_2 are positive constants depending only on Ω and on the differential operators L and R_μ .

Theorem 3. Suppose that condition 1) of Theorem 2 is satisfied and $l \geq 2k$. Then, for every $u \in W_p^{(l)}(\Omega)$ satisfying the boundary conditions

$$R_\mu u|_S = 0 \quad (\mu = 1, 2, \dots, k),$$

the inequality

$$C_1 \|Lu\|_{W_p^{(l-2k)}(\Omega)} \leq \|u\|_{W_p^{(l)}(\Omega)} \leq C_2 \left[\|Lu\|_{W_p^{(l-2k)}(\Omega)} + \|u\|_{L_p(\Omega)} \right]. \quad (4)$$

Theorems 2 and 3 generalize the corresponding results of O. V. Guseva ⁽⁴⁾, F. Browder ⁽⁵⁾, A. I. Koshelev ⁽⁶⁾, and L. Nirenberg ^(7,8).

3. Let us now consider, in the cylinder $Q = \Omega \times [0, T]$ ($x \in \Omega$; $0 \leq t \leq T \leq +\infty$), the parabolic differential operator

$$L\left(t, x, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right)u = \frac{\partial u}{\partial t} - \sum_{r=1}^{2k} \sum_{i_1, \dots, i_r=1}^n A^{(i_1, \dots, i_r)}(t, x) \frac{\partial^r u}{\partial x_{i_1} \dots \partial x_{i_r}} - A(t, x)u.$$

The operator

$$L_0\left(t, x, \frac{\partial}{\partial t}, \frac{\partial}{\partial x}\right) = \frac{\partial}{\partial t} - \sum_{i_1, \dots, i_{2k}=1}^n A^{(i_1, \dots, i_{2k})}(t, x) \frac{\partial^{2k}}{\partial x_{i_1} \dots \partial x_{i_{2k}}}$$

will be called the principal part of the operator L .

On $\Gamma = S \times [0, T]$ let k linear differential operators be given:

$$R_\mu(t, x', \partial/\partial t, \partial/\partial x) \quad (\mu = 1, 2, \dots, k).$$

We shall call the principal part of R_μ the collection of all its terms of the form $B(t, x')D_x^{s_1}D_t^{s_2}$, for which the quantity $s_1 + 2ks_2$ is greatest. In this case

$$m_\mu = \max\{s_1 + 2ks_2\}$$

will be called the order of R_μ .

Proceeding as before, we transform L and R_μ to the variables $t, \gamma_1, \dots, \gamma_n$. Denote the results of the transformation by

$$L(t, \gamma, \partial/\partial t, \partial/\partial \gamma)$$

and

$$R_\mu(t, \gamma', \partial/\partial t, \partial/\partial \gamma).$$

Introduce the matrix

$$D(t, \gamma', \beta_0, \beta') = \left\| \int_{C(\beta_0, \beta')} R_\mu^{(0)}(t, \gamma', i\beta_0, i\beta) L_0^{-1}(t, \gamma', i\beta_0, i\beta) \beta_n^{\lambda-1} d\beta_n \right\|_{\mu, \lambda=1, 2, \dots, k},$$

where $C(\beta_0, \beta')$ is a positively oriented contour lying in the upper half-plane β_n and enclosing all the roots situated there of the equation

$$|L_0(t, \gamma', i\beta_0, i\beta)| = 0$$

for fixed t, γ', β_0 , and β' .

We shall say that L and R_μ ($\mu = 1, 2, \dots, k$) are connected on Γ by condition $(\mathcal{L})$, if for every point $(t, x') \in \Gamma$ and any real $\beta_0, \beta_1, \dots, \beta_{n-1}$ for which $\beta_0^2 + |\beta'|^2 = 1$, the inequality

$$\|D(t, \gamma', \beta_0, \beta')\| \geq \delta' \quad (\delta' > 0)$$

holds.

Theorem 4. Let: 1) $l_1 = 2kl_2$, $l_2 \geq 1$, L be a parabolic differential operator, and let L and R_μ ($\mu = 1, 2, \dots, k$) be connected on Γ by condition $(\mathcal{L})$; 2) $m_\mu < l_1 - \frac{1}{2}$ ($\mu = 1, 2, \dots, k$). Then for any vector-function $u \in W_{x,t,2}^{(l_1, l_2)}(Q)$ the inequality

$$C_1 \left[\|Lu\|_{W_{x,t,2}^{(l_1-2k, l_2-1)}(Q)} + \|u\|_{W_2^{(l)}(\Omega)} + \sum_{\mu=1}^k \|R_\mu u\|_{W_{x,t,2}^{(l_\mu^{(1)}, l_\mu^{(2)})}(\Gamma)} \right] \leq \|u\|_{W_{x,t,2}^{(l_1, l_2)}(Q)} \leq C_2 \left[\|Lu\|_{W_{x,t,2}^{(l_1-2k, l_2-1)}(Q)} + \|u\|_{L_p(Q)} \right]$$

where

$$l = l_1 \left(1 - \frac{1}{2l_2} \right) = l_1 - k = (2l_2 - 1)k, \quad l_\mu^{(1)} = l_1 - m_\mu - \frac{1}{2},$$

$$l_\mu^{(2)} = l_2 \left(1 - \frac{m_\mu}{l_1} - \frac{1}{2l_1} \right) \quad (\mu = 1, 2, \dots, k),$$

and C_1 and C_2 are positive constants depending only on Q , L , and R_μ ($\mu = 1, 2, \dots, k$).

Theorem 5. Suppose that condition 1) of Theorem 4 is satisfied. Then for any $u \in W_{x,t,p}^{(l_1, l_2)}(Q)$ satisfying the conditions $u|_{t=0} = R_\mu u|_\Gamma = 0$ ($\mu = 1, 2, \dots, k$), the inequality

$$C_1 \|Lu\|_{W_{x,t,p}^{(l_1-2k, l_2-1)}(Q)} \leq \|u\|_{W_{x,t,p}^{(l_1, l_2)}(Q)} \leq C_2 \left[\|Lu\|_{W_{x,t,p}^{(l_1-2k, l_2-1)}(Q)} + \|u\|_{L_p(Q)} \right]. \quad (6)$$

Theorems 4 and 5 admit generalization to arbitrary parabolic systems of I. G. Petrovskii ⁽⁹⁾ with derivatives with respect to t of arbitrary orders. These theorems generalize results of O. A. Ladyzhenskaya ⁽¹⁰⁾.

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Received
16 I 1958

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Note: Figure translations are in progress. See original paper for figures.

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