

# ON THE STRUCTURE OF SOLUTIONS OF AUTOMORPHIC SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

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**Abstract**

**Full Text**

**MATHEMATICS**

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# ON THE STRUCTURE OF SOLUTIONS OF AUTOMORPHIC SYSTEMS OF LINEAR DIFFERENTIAL EQUATIONS

*(Presented by Academician I. G. Petrovskii, 3 January 1958)*

We shall call a system of differential equations

$$\frac{dx_k}{dz} = \sum_{l=1}^n P_{kl}(z)x_l, \quad k = 1, 2, \dots, n, \quad (1)$$

with meromorphic coefficients  $P_{kl}$  **automorphic** if it does not change under the fractional-linear transformation

$$z' = T(z) = \frac{ez + f}{gz + h}, \quad eh - fg = 1. \quad (2)$$

A particular case of automorphic systems is furnished by systems with periodic and doubly periodic coefficients. The construction of automorphic systems with many substitutions was carried out in the work <sup>(1)</sup>. Here we shall clarify the structure of the solutions of system (1), invariant with respect to the substitution (2).

The structure of an automorphic system and of its solutions depends essentially on the fixed points of the linear transformation (2), i.e., on the roots of the equation

$$gz^2 + (h - e)z - f = 0. \quad (3)$$

It is necessary to distinguish three cases: 1) both fixed points lie at a finite distance; 2) one fixed point is at the infinitely distant point; 3) both fixed points are infinitely distant. In the first case the transformation (2) is a proper fractional-linear transformation ( $g \neq 0$ ); in the second case it will be of the form  $sz + \Lambda$ , with  $s \neq 1$ ; and in the third case  $T = z + \Omega$ . The third case is well studied: here there are theorems of Floquet (for equations with periodic coefficients) and of Picard (for equations with doubly periodic coefficients). The first two cases have not been considered in the literature, but precisely these cases are of the greatest interest for automorphic systems.

§ 1. Let us first clarify the general structure of the solutions of equations (1) in connection with their invariance with respect to the substitution (2). Let the functions

$$x_k^j(z), \quad j, k = 1, 2, \dots, n, \quad (4)$$

be fundamental solutions of system (1), satisfying the initial conditions  $x_k^j(0) = \delta_{jk}$ , where  $\delta_{jk}$  is the Kronecker symbol. The functions  $x_k^j(T(z))$  will also be solutions. Consequently,

$$x_k^j(T(z)) = \sum_{l=1}^n a_l^j x_k^l(z). \quad (5)$$

where  $a_k^j = x_k^j(T(0))$ . For brevity, let us pass to matrix notation: introduce the matrices

$$X = \|x_k^j\|_1^n, \quad A = \|a_k^j\|_1^n$$

and write the last equality in the form  $X(T) = AX(z)$ . Put

$$Y(z) = BX(z), \quad (6)$$

and choose the matrix  $B$  so that the matrix  $BAB^{-1}$  assumes Jordan canonical form. If the roots of the characteristic equation  $|A - \rho I| = 0$  are simple, then

$$BAB^{-1} = \left\| \begin{array}{cccc} \rho_1 & 0 & \dots & 0 \\ 0 & \rho_2 & \dots & 0 \\ \cdot & \cdot & \dots & \cdot \\ 0 & 0 & \dots & \rho_n \end{array} \right\|, \quad (7)$$

and the relations (5) take the form

$$y_k^j(T(z)) = \rho_j y_k^j(z). \quad (8)$$

In the case of multiple roots, after separating the elementary divisors we shall have

$$BAB^{-1} = [J_{q_1}(\xi_1), \dots, J_{q_m}(\xi_m)], \quad (9)$$

$$J_q(\xi) = \begin{vmatrix} \xi & 0 & 0 & \cdots & 0 & 0 \\ 1 & \xi & 0 & \cdots & 0 & 0 \\ 0 & 1 & \xi & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & \xi \end{vmatrix}.$$

The relations (5) will be written in the form

$$\begin{aligned} y_k^1(T) &= \xi_1 y_k^1(z), \\ y_k^2(T) &= \xi_1 y_k^2(z) + y_k^1(z), \\ &\dots \\ y_k^{q_1}(T) &= \xi_1 y_k^{q_1}(z) + y_k^{q_1-1}(z), \\ &\dots \\ y_k^{n-q_m-1+1}(T) &= \xi_m y_k^{n-q_m-1+1}(z), \\ y_k^{n-q_m-1+2}(T) &= \xi_m y_k^{n-q_m-1+2}(z) + y_k^{n-q_m-1+1}(z), \\ &\dots \\ y_k^n(T) &= \xi_m y_k^n(z) + y_k^{n-1}. \end{aligned} \tag{10}$$

§ 2. Let the function  $\sigma(z)$  possess the following property:

$$\sigma(T(z)) = \sigma(z) + 1 \tag{11}$$

(later we shall construct this function explicitly). In the case of simple roots of the characteristic equation, consider the functions

$$u_k^j(z) = \frac{y_k^j(z)}{\rho_k^{\sigma(z)}}. \tag{12}$$

By virtue of (8) and (12) we have:  $u_k^j(T) = u_k^j(z)$ , i.e., the function  $u_k^j(z)$  possesses the property of automorphy with respect to the substitution (2), and, if it is meromorphic, then this relation is an automorphic function. Returning to the original unknowns, we obtain the solutions of system (1) in the form

$$x_k(z) = \sum_{j=1}^n \rho_j^{\sigma(z)} \alpha_{kj}(z), \tag{13}$$

where  $\alpha_{kj}(z)$  are invariant with respect to the substitution (2).

In the case of multiple elementary divisors, construct the functions

$$\sigma_0 = 1, \quad \sigma_1 = \sigma(z), \quad \sigma_2 = \frac{\sigma(\sigma-1)}{2!}, \quad \sigma_3 = \frac{\sigma(\sigma-1)(\sigma-2)}{3!}, \dots, \quad (14)$$

which have the following properties:

$$\sigma_{i+1}(T(z)) = \sigma_{i+1}(z) + \sigma_i(z). \quad (15)$$

Next construct the functions

$$\begin{aligned} \beta_1 &= \xi_1^{\sigma_1(z)} \alpha_1(z), \\ \beta_2 &= \xi_1^{\sigma_1(z)} \left[ \alpha_2(z) + \frac{\sigma_1(z)}{\xi_1} \alpha_1(z) \right], \\ \beta_3 &= \xi_1^{\sigma_1(z)} \left[ \alpha_3(z) + \frac{\sigma_1(z)}{\xi_1} \alpha_2(z) + \frac{\sigma_2(z)}{\xi_1^2} \alpha_1(z) \right], \\ &\dots\dots\dots \\ \beta_{q_1} &= \xi_1^{\sigma_1(z)} \left[ \alpha_{q_1}(z) + \frac{\sigma_1(z)}{\xi_1} \alpha_{q_1-1}(z) + \dots + \frac{\sigma_{q_1-1}(z)}{\xi_1^{q_1-1}} \alpha_1(z) \right] \end{aligned} \quad (16)$$

( $\alpha_i(z)$  are invariant with respect to the substitution under consideration) and analogous functions for the remaining elementary divisors. From the structure of the functions  $\beta_i$  it is clear that

$$\begin{aligned} \beta_1(T(z)) &= \xi_1 \beta_1(z), \\ \beta_2(T(z)) &= \xi_1 \beta_2(z) + \beta_1(z), \\ &\dots\dots\dots \\ \beta_{q_1}(T(z)) &= \xi_1 \beta_{q_1}(z) + \beta_{q_1-1}(z). \end{aligned} \quad (17)$$

Comparing the last equalities with (10), we see that

$$\begin{aligned} y_k^1(z) &= \xi_1^{\sigma_1} \alpha_{k1}, \\ y_k^2(z) &= \xi_1^{\sigma_1} \left[ \alpha_{k2} + \frac{\sigma_1}{\xi_1} \alpha_{k1} \right], \\ &\dots\dots\dots \\ y_k^{q_1}(z) &= \xi_1^{\sigma_1} \left[ \alpha_{kq_1} + \frac{\sigma_1}{\xi_1} \alpha_{kq_1-1} + \dots + \frac{\sigma_{q_1-1}}{\xi_1^{q_1-1}} \alpha_{k1} \right], \end{aligned} \quad (18)$$

where  $\alpha_{kj}$  are invariant with respect to the substitution (2).

§ 3. It remains to show that the automorphic systems under consideration exist, and to construct for them the function  $\sigma$ .

Let  $T = sz + \Lambda$ . The function  $\mu(z) = z - \frac{\Lambda}{1-s}$  has the property:  $\mu(T) = s\mu(z)$ . If  $U_{kl}$  are automorphic functions with respect to the substitution under consideration, then the system of differential equations

$$\mu(z) \frac{dx_k}{dz} = \sum_{l=1}^n U_{kl}(z) x_l \quad (19)$$

$\left(\frac{\Lambda}{1-s} = \lambda \text{ is a fixed point}\right)$  will be automorphic. Thus, for example, for the substitution  $-z + 1$  we have the automorphic system

$$\left(z - \frac{1}{2}\right) \frac{dx_k}{dz} = \sum_{l=1}^n R_{kl} \left(z, \frac{1}{z}, 1-z, \frac{1}{1-z}, \frac{z-1}{z}, \frac{z}{z-1}\right) x_l.$$

The function  $\sigma$  for a substitution with one finite fixed point has the form:

$$\sigma(z) = \frac{\ln(z - \lambda)}{\ln s}. \quad (20)$$

In the case of two finite fixed points we construct the function

$$\nu(z) = (z - \lambda_1)(z - \lambda_2),$$

$$\lambda_{1,2} = \frac{1}{2g} \left( e - h \pm \sqrt{(e - h)^2 + 4fg} \right). \quad (21)$$

From the structure of the function  $\nu(z)$  it is seen that  $\nu(T) = \nu(z)/(gz + h)^2$ . In accordance with this, we find the automorphic system

$$\nu(z) \frac{dx_k}{dz} = \sum_{l=1}^n U_{kl}(z) x_l, \quad (22)$$

where  $U_{kl}$  are automorphic functions with the given substitution. Simple calculations show that, for the solutions of the automorphic system (22), the function  $\sigma$  has the form:

$$\sigma(z) = \frac{\ln \frac{z - \lambda_1}{z - \lambda_2}}{\ln \frac{e - \lambda_1 g}{e - \lambda_2 g}}. \quad (23)$$

Thus, the question of the structure both of automorphic systems and of their solutions is constructively resolved completely.

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## CITED LITERATURE

1. O. I. Samuil, *Bull. Central Asian State Univ.*, issue 30 (1949).

*Note: Figure translations are in progress. See original paper for figures.*

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