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Abstract

Full Text

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MATHEMATICS

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ON THE BEST APPROXIMATION OF CLASSES OF FUNCTIONS REPRESENTABLE IN THE FORM OF A CONVOLUTION

(Presented by Academician N. N. Bogolyubov on 29 VI 1957)

1. Let H_p ($1 \leq p < \infty$) be the class of real functions $\varphi(t) \in L^p$ with period ω , satisfying the condition

$$\|\varphi\| = \left\{ \int_0^\omega |\varphi(t)|^p dt \right\}^{1/p} \leq 1,$$

and let H_∞ be the class of essentially bounded measurable functions $\varphi(t)$ with period ω , for which

$$\text{ess sup } |\varphi(t)| \leq 1.$$

Suppose that the function $f(x)$ admits a representation in the form of a convolution:

$$f(x) = \frac{1}{\omega} \int_0^\omega K(t-x) \varphi(t) dt \quad (\varphi(t) \in H_p), \quad (1)$$

where $K(t)$ is a function with period ω , and, in the case $1 < p \leq \infty$, $K(t) \in L^q$

$$\left(\frac{1}{p} + \frac{1}{q} = 1 \right),$$

while in the case $p = 1$, $K(t)$ is continuous. The totality of all functions of the form (1) will be denoted by $C(K, H_p)$.

Here we consider the problem of the best approximation of the class of functions $f(x)$, representable in the form (1), by polynomials in a given Chebyshev system $(^1)$ of continuous functions with period ω , $\{f_k(x)\}$ ($k = 1, 2, \dots, n$); general formulas are established for the quantity

$$M_n^{(p)} = \sup_{\varphi \in H_p} E_n(f)_C, \quad 1 \leq p \leq \infty, \quad (2)$$

where

$$E_n(f)_C = \min_{\alpha_k} \max_{0 \leq x \leq \omega} \left| f(x) - \sum_{k=1}^n \alpha_k f_k(x) \right|$$

(α_k are real numbers) is the best approximation of the function $f(x)$ by polynomials in the system $\{f_k(x)\}$; the properties of extremal functions $f_0(x)$, for which $E_n(f_0)_C = M_n^{(p)}$, are investigated, and the relations between the quantity $M_n^{(p)}$ and the best approximation of the corresponding kernel $K(t)$ in the metric L^q are studied.

For the case $p = \infty$, this problem was studied earlier by Favard ⁽²⁾, Akhiezer and Krein ⁽³⁾, Nadem ⁽⁴⁾, Nikol'skii ⁽⁵⁾, and others.

2. Let

$$0 \leq x_1 < x_2 < \dots < x_{n+1} < \omega.$$

Put ⁽⁶⁾

$$M_i = \begin{vmatrix} f_1(x_1) & f_1(x_2) & \dots & f_1(x_{i-1}) & f_1(x_{i+1}) & \dots & f_1(x_{n+1}) \\ f_2(x_1) & f_2(x_2) & \dots & f_2(x_{i-1}) & f_2(x_{i+1}) & \dots & f_2(x_{n+1}) \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ f_n(x_1) & f_n(x_2) & \dots & f_n(x_{i-1}) & f_n(x_{i+1}) & \dots & f_n(x_{n+1}) \end{vmatrix},$$

$$M = \sum_{i=1}^{n+1} M_i, \quad a_i = (-1)^i \frac{M_i}{M} \quad (i = 1, 2, \dots, n+1).$$

By $a_i^{(0)}$ we shall denote the value of a_i for $x_j = x_j^{(0)}$ ($j = 1, 2, \dots, n+1$), and by $\{x_i^{(0)}\}$ a system of points

$$0 \leq x_1^{(0)} < x_2^{(0)} < \dots < x_{n+1}^{(0)} < \omega.$$

In what follows, by a Chebyshev alternant of the function $f_0(x)$ we shall mean a system of points $\{x_i^{(0)}\}$ at which the difference between the given function and its polynomial of best approximation assumes the maximal values $E_n(f_0)_C$ with alternating signs.

3. The case $1 < p \leq \infty$.

Theorem 1. Let $K(t) \in L^q \left(\frac{1}{p} + \frac{1}{q} = 1 \right)$.

Then:

$$1^\circ. \quad M_n^{(p)} = \frac{1}{\omega} \max_{0 < x_1 < \dots < x_{n+1} < \omega} \left\{ \int_0^\omega \left| \sum_{i=1}^{n+1} a_i K(t - x_i) \right|^q dt \right\}^{1/q}. \quad (3)$$

2°. There exists a function $f_0(x) \in C(K, H_p)$ for which $E_n(f_0)_C = M_n^{(p)}$.

3°. The maximum in formula (3) is attained for those and only those systems of points $\{x_i^{(0)}\}$ which are Che

Generally speaking, the extremal function $f_0(x)$ is not unique. In the case $1 < p < \infty$, each extremal function $f_0(x)$ is determined by a unique function $\varphi_0(t) \in H_p$. In order that, for $M_n^{(p)} > 0$, the function $\varphi_0(t) \in H_p$ determine an extremal function $f_0(t)$, it is necessary and sufficient that there exist a system of points $\{x_i^{(0)}\}$ such that

$$M_n^{(p)} = \frac{1}{\omega} \left\{ \int_0^\omega \left| \sum_{i=1}^{n+1} a_i^{(0)} K(t - x_i^{(0)}) \right|^q dt \right\}^{1/q}$$

and, almost everywhere,

$$\varphi_0(t) = \left\{ \frac{1}{\omega M_n^{(p)}} \right\}^{q-1} \left| \sum_{i=1}^{n+1} a_i^{(0)} K(t - x_i^{(0)}) \right|^{q-1} \operatorname{sign} \left\{ \sum_{i=1}^{n+1} a_i^{(0)} K(t - x_i^{(0)}) \right\}$$

on the interval $0 \leq t \leq \omega$.

The extremal function $f_0(x)$ is unique if and only if all the functions

$$\left| \sum_{i=1}^{n+1} a_i^{(0)} K(t - x_i^{(0)}) \right| \operatorname{sign} \left\{ \sum_{i=1}^{n+1} a_i^{(0)} K(t - x_i^{(0)}) \right\}$$

coincide almost everywhere on $[0, \omega)$ for all systems of points $\{x_i^{(0)}\}$ for which the maximum in formula (3) is attained.

In the case $p = \infty$, one and the same extremal function $f_0(x)$ may be determined by different functions $\varphi_0(t) \in H_\infty$. In order that the function $\varphi_0(t) \in H_\infty$ determine an extremal function $f_0(x)$, it is necessary and sufficient that there exist a system of points $\{x_i^{(0)}\}$ such that

$$M_n^{(\infty)} = \frac{1}{\omega} \int_0^\omega \left| \sum_{i=1}^{n+1} a_i^{(0)} K(t - x_i^{(0)}) \right| dt,$$

$$\varphi_0(t) = \operatorname{sign} \left\{ \sum_{i=1}^{n+1} a_i^{(0)} K(t - x_i^{(0)}) \right\}$$

almost everywhere on the set

$$\mathcal{E} \left\{ t : \sum_{i=1}^{n+1} a_i^{(0)} K(t - x_i^{(0)}) \neq 0 \right\}, \quad \text{mes } \mathcal{E} > 0.$$

For there to exist a unique function $\varphi_0(t) \in H_\infty$ determining the extremal function $f_0(x)$, it is necessary and sufficient that all the functions

$$\text{sign} \left\{ \sum_{i=1}^{n+1} a_i^{(0)} K(t - x_i^{(0)}) \right\}$$

coincide almost everywhere on $[0, \omega)$ for all systems of points $\{x_i^{(0)}\}$ for which the maximum in formula (3) is attained.

In the following two theorems we shall assume that $K(t) \in L^q[0, 2\pi)$ ($1 \leq q < \infty$); $\varphi(t) \in H_p[0, 2\pi)$ ($\frac{1}{p} + \frac{1}{q} = 1$); $T_n(t)$ is a trigonometric polynomial of degree not exceeding n , and

$$E_n(f)_c = \min_{T_{n-1}} \|f(x) - T_{n-1}(x)\|_c,$$

where $f(x)$ is an arbitrary continuous function with period 2π .

Theorem 2. Let $K(t) \in L^q[0, 2\pi)$ ($1 \leq q < \infty$) and

$$f(x) = \frac{1}{\pi} \int_0^{2\pi} K(t - x) \varphi(t) dt, \quad (4)$$

where $\varphi(t) \in H_p[0, 2\pi)$. Then

$$\sup_{\varphi \in H_p} E_n(f)_c \leq \frac{1}{\pi} \min_{T_{n-1}} \left\{ \int_0^{2\pi} |K(t) - T_{n-1}(t)|^q dt \right\}^{1/q}, \quad (5)$$

where

$$\frac{1}{p} + \frac{1}{q} = 1.$$

If $1 < p < \infty$, then in order that inequality (5) become an equality, it is necessary and sufficient that there exist a system of points

$$0 \leq x_1^{(0)} < \dots < x_{2n}^{(0)} < 2\pi$$

such that:

$$1^\circ. \quad \sup_{\varphi \in H_p} E_n(f)_c = \frac{1}{\pi} \left\{ \int_0^{2\pi} \left| \sum_{i=1}^{2n} \tilde{a}_i^{(0)} K(t - x_i^{(0)}) \right|^q dt \right\}^{1/q}, \quad (6)$$

where

$$\tilde{a}_i^{(0)} = (-1)^i \frac{Q_i^{(0)}}{Q^{(0)}}, \quad Q^{(0)} = \sum_{i=1}^{2n} Q_i^{(0)}, \quad Q_i^{(0)} = \prod_{p,q \neq i} \sin \frac{x_p^{(0)} - x_q^{(0)}}{2}, \quad (7)$$

$$2n \geq p > q \geq 1, \quad p, q \neq i, \quad i = 1, 2, \dots, 2n.$$

$$2^\circ. \quad \text{sign}\{K_n(t - x_i^{(0)})\} \cdot \text{sign}\{K_n(t - x_{i+1}^{(0)})\} \leq 0$$

almost everywhere on the interval $0 \leq t \leq 2\pi$, $i = 1, 2, \dots, 2n - 1$.

3°. All the functions $|K_n(t - x_i^{(0)})|$ are proportional on the interval $0 \leq t \leq 2\pi$, $i = 1, 2, \dots, 2n$, where

$$K_n(t) = K(t) - T_{n-1}^*(t),$$

and $T_{n-1}^*(t)$ is the polynomial of best approximation to the function $K(t)$ of degree $n - 1$ in L^q .

If, however, $p = \infty$, then for (5) to become an equality it is necessary that there exist a system of points

$$0 \leq x_1^{(0)} < x_2^{(0)} < \dots < \dots < x_{2n}^{(0)} < 2\pi$$

such that:

$$1^\circ. \quad \sup_{\varphi \in H_\infty} E_n(f)_c = \frac{1}{\pi} \int_0^{2\pi} \left| \sum_{i=1}^{2n} \tilde{a}_i^{(0)} K(t - x_i^{(0)}) \right| dt. \quad (8)$$

$$2^\circ. \quad \text{sign}\{K_n(t - x_i^{(0)})\} \cdot \text{sign}\{K_n(t - x_{i+1}^{(0)})\} \leq 0$$

almost everywhere on the interval $0 \leq t \leq 2\pi$, $i = 1, 2, \dots, 2n - 1$, where $K_n(t) = K(t) - T_{n-1}^*(t)$, and $T_{n-1}^*(t)$ is any polynomial of best approximation in the mean to the function $K(t)$ of order $n - 1$.

If these conditions are satisfied for some trigonometric polynomial $T_{n-1}^*(t)$ of best approximation in the mean to the function $K(t)$ of order $n - 1$, then (5) turns into the equality (5').

Theorem 3. If $K(t) \in L$, $K(t + 2\pi) = K(t)$, $\varphi(t)$ is any continuous function with period 2π ,

$$f(x) = \frac{1}{\pi} \int_0^{2\pi} K(t - x) \varphi(t) dt, \quad (9)$$

then

$$E_n(f)_C \leq \frac{1}{\pi} \left\{ \max_{0 < x_1 < \dots < x_{2n} < 2\pi} \int_0^{2\pi} \left| \sum_{i=1}^{2n} \tilde{a}_i K(t - x_i) \right| dt \right\} E_n(\varphi)_C, \quad (10)$$

where $\tilde{a}_i$ is a function of the parameters x_j ($1 \leq j \leq 2n$, $j \neq i$), defined by (7).

The inequality (10) cannot be improved.

Corollary. Let $f(x)$ be a periodic function with period 2π , having a continuous r -th derivative $f^{(r)}(x)$; then

$$E_n(f)_C \leq \frac{K_r}{n^r} E_n(f^{(r)})_C, \quad n = 1, 2, \dots \quad (11)$$

If, moreover, the conjugate function $\tilde{f}(x)$ has a continuous r -th derivative $\tilde{f}^{(r)}(x)$, then

$$E_n(f)_C \leq \frac{\tilde{K}_r}{n^r} E_n(\tilde{f}^{(r)})_C, \quad n = 1, 2, \dots \quad (12)$$

Here K_r , $\tilde{K}_r$ are the Favard constants (^{3,7}). The constants cannot be improved.

4. We now consider the case $p = 1$, $K(t) \in C[0, \omega]$. Let H_V be the class of functions $g(t)$ of bounded variation on $[0, \omega]$ with norm $\|g\|_V = \int_0^\omega |dg| \leq 1$. Put

$$F(x) = \frac{1}{\omega} \int_0^\omega K(t - x) dg(t). \quad (13)$$

Theorem 4. If $K(t) \in C[0, \omega]$, then

$$M_n^{(1)} = \max_{g \in H_V} E_n(F)_C = \frac{1}{\omega} E_n(K)_C. \quad (14)$$

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Note: Figure translations are in progress. See original paper for figures.

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