

# MATHEMATICS

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**Abstract**

**Full Text**

MATHEMATICS

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# ON THE CONNECTION BETWEEN A LINEAR DYNAMICAL SYSTEM AND A DIFFERENTIAL EQUATION IN A BANACH SPACE

*(Presented by Academician I. G. Petrovskii, 11 I 1958)*

The qualitative theory of differential equations given in a Banach space  $B$  has recently been considered by a number of authors (<sup>1-3</sup>). In some works (for example, (<sup>1-3</sup>)) separate questions connected with stability, boundedness, and the rate of decrease or growth of solutions of the equation  $dp/dt = A(t)p$ ,  $p \in B$ , are investigated. The linear operator  $A(t)$ , as a rule, is assumed to be bounded for each fixed  $t$ . M. G. Krein, in particular, considers the equation

$$dp/dt = Ap, \quad (1)$$

where the linear bounded operator  $A$  does not depend on  $t$ .

At the same time, in the qualitative theory of differential equations an important place is occupied by the theory of dynamical systems, in which the properties of integral curves are studied without reference to differential equations.

The dynamical system  $f(p, t) = f(t)p$  (<sup>4</sup>), p. 347), corresponding to equation (1), satisfies the usual axioms:

A1.  $f(p, 0) = p$ , i.e.  $f(0) = I$  (the identity operator).

A2.  $f[f(p, t_1), t_2] = f(p, t_1 + t_2)$ .

A3.  $f(p, t)$  is continuous in the aggregate of its arguments.

In addition to these axioms, it also possesses the property of linearity:

A4.  $f(p_1 + p_2, t) = f(p_1, t) + f(p_2, t)$  (this is easy to verify by solving equation (1) by the method of successive approximations).

The problem naturally arises of studying a linear dynamical system, i.e. a system satisfying conditions A1–A4 (the 1st problem). The problem in this form will have meaning and interest if the indicated formulation proves to be more general than the qualitative study of equation (1) (the 2nd problem)\*.

In the present note it will be shown that in the case of a finite-dimensional space  $B$  the 1st and 2nd problems are equivalent, but for an infinite-dimensional space

the 1st problem is more general than the 2nd. In addition, one question posed by E. Hille <sup>(5)</sup> will be solved.

In what follows, the term “linear operator” is used in the sense of “bounded linear operator.”

**Theorem 1.** In order that the linear dynamical system  $f(t)p$ , given in a Banach space  $B$ , correspond to equation (1), it is necessary and sufficient that

$$\lim_{t \rightarrow 0} \|f(t) - I\| = 0.$$

(In other words, the operator function  $f(t)$  is uniformly continuous <sup>(5)</sup>, Definition 3.4.2) at  $t = 0$ .)

\* The formulation of the 1st problem is due to E. A. Barbashin.

**Proof. Necessity.** By assumption, at each point  $p \in B$  there exists a derivative along the trajectory

$$\frac{dp}{dt} = \lim_{t \rightarrow 0} \frac{f(t)p - p}{t} = \lim_{t \rightarrow 0} \frac{1}{t} [f(t) - I]p = Ap.$$

Consider the family of linear operators  $\{F(t)\}$ , where

$$F(t) = \frac{1}{t} [f(t) - I], \quad -1 \leq t < 0 \text{ or } 0 < t \leq 1.$$

Since  $\lim_{t \rightarrow 0} F(t)p = Ap$ , it follows that for any point  $p \in B$ ,

$$\sup_t \|F(t)p\| < \infty.$$

By Banach's theorem (<sup>(5)</sup>, Theorem 2.12.2), it follows that the set  $\{\|F(t)\|\}$  is bounded. But

$$\|F(t)\| = \frac{\|f(t) - I\|}{|t|},$$

and, as  $t \rightarrow 0$ , the boundedness of  $\|F(t)\|$  means that  $\|f(t) - I\| \rightarrow 0$ .

**Sufficiency.** Let  $\|f(t) - I\| \rightarrow 0$ . To prove this part of the theorem it is enough to note that  $f(t)$  is a function defined on the interval  $(-\infty, \infty)$ , with values in the Banach algebra of endomorphisms of the space  $B$  into itself (<sup>(5)</sup>, Definition 2.15.1 and Theorem 2.15.2).

Since  $f(t_1 + t_2) = f(t_1) \cdot f(t_2)$  (condition  $A_2$ ) and there exists the uniform limit  $\lim_{t \rightarrow 0} f(t) = I$ , all the conditions of Theorem 8.4.2 from <sup>(5)</sup> are satisfied, and consequently there exists (in the uniform topology)

$$\lim_{t \rightarrow 0} \frac{1}{t} [f(t) - I] = A.$$

The operator  $A$ , of course, is linear (from the proof of the theorem it is clear that passage to the limit is free from the condition  $t > 0$ ). Then

$$\frac{dp}{dt} = \lim_{t \rightarrow 0} \frac{1}{t} [f(t) - I]p = Ap.$$

$$\frac{dp}{dt} = Ap$$

is the desired differential equation. Since the solution of this equation is unique, the corresponding dynamical system coincides with the system  $f(t)p$ .

**Theorem 2.** *In order that a one-parameter group  $f(t)$  of linear operators define a dynamical system  $f(t)p$ , it is sufficient that two conditions be fulfilled: 1)  $f(0) = I$ ; 2)  $f(t)$  is strongly continuous at  $t = 0$  (the necessity of these conditions is obvious).*

**Proof.** It is only required to establish the continuity of the function  $f(t)p$  in the aggregate of its arguments. The equality

$$\lim_{t \rightarrow t_0} f(t)p = \lim_{t \rightarrow t_0} f(t - t_0)f(t_0)p = f(t_0)p$$

shows the strong continuity of  $f(t)$  on the entire axis. Hence follows the continuity of  $\|f(t)p\|$  ((<sup>5</sup>), p. 31). If now  $a \leq t \leq b$ , then for any point  $p \in B$ ,

$$\sup_t \|f(t)p\| < \infty,$$

and, by Banach's theorem ((<sup>5</sup>), Theorem 2.12.2), the set  $\{\|f(t)\|\}$  is bounded by a number  $M$  depending only on  $a$  and  $b$ . Using the estimate

$$\|f(t)p - f(t_0)p_0\| \leq \|f(t)\| \cdot \|p - p_0\| + \|f(t)p_0 - f(t_0)p_0\|,$$

it is easy to finish the proof. Examples can be given showing that conditions 1) and 2) do not follow from the other conditions of the theorem.

**Theorem 3.** *If a one-parameter group of linear operators  $f(t)$ , acting in a finite-dimensional space, is given, then from the strong continuity of  $f(t)$  at  $t = 0$  there follows the uniform continuity of  $f(t)$  at  $t = 0$ .*

**Proof.** Suppose the contrary. Since

$$\|f(t) - f(0)\| = \sup_{\|p\| \leq 1} \|(f(t) - f(0))p\|,$$

the supposition means that there exists a number  $a > 0$ , a numerical sequence

$$t_1, t_2, \dots, t_n, \dots \rightarrow 0$$

and a sequence of points of the space

$$p_1, p_2, \dots, p_n, \dots, \quad \|p_n\| \leq 1,$$

such that, for every  $n$ , the inequality

$$\|f(t_n)p_n - f(0)p_n\| \geq a$$

holds. Since in a finite-dimensional space the sphere  $\|p\| \leq 1$  is compact, we may suppose that

$$\lim p_n = q.$$

In Theorem 2 it was proved that the strong continuity of  $f(t)$  entails the continuity of the function  $f(t)p$  in the aggregate.

arguments. Passing to the limit in the inequality  $\|f(t_n)p_n - f(0)p_n\| \geq a$  as  $n \rightarrow \infty$ , we arrive at a contradiction:  $\|f(0)q - f(0)q\| \geq a$ .

**Corollary.** In a finite-dimensional space, a linear dynamical system always corresponds to a differential equation.

For the proof it suffices to compare Theorems 1 and 3.

**Definition.** Two linear dynamical systems  $f(t)p$  and  $g(t)q$ ,  $p \in B$ ,  $q \in B'$ , will be called **linearly homeomorphic** if there exists a linear bicontinuous isomorphism  $\Phi$  of the Banach spaces  $B$  and  $B'$  such that from the equality  $q = \Phi(p)$  there follows the equality  $g(t)q = \Phi[f(t)p]$ .

**Theorem 4.** If a linear dynamical system  $f(t)p$  corresponds to the differential equation (1), then every dynamical system  $g(t)q$  linearly homeomorphic to the system  $f(t)p$  has the same property.

**Proof.** Let  $q = \Phi(p)$ ,  $g(t)q = \Phi \cdot f(t)p$ . But  $p = \Phi^{-1}(q)$ , therefore  $g(t) = \Phi \cdot f(t) \cdot \Phi^{-1}$ ,  $\|g(0) - g(t)\| = \|\Phi[f(0) - f(t)]\Phi^{-1}\| \leq \|\Phi\| \cdot \|f(0) - f(t)\| \cdot \|\Phi^{-1}\|$ . The validity of the theorem is now obvious.

E. Hille ((<sup>5</sup>), p. 225) notes: "Whether  $\|T(\xi)\|$  is always continuous is unknown." In our notation,  $T(\xi)$  is  $f(t)$  for  $t > 0$  (a semigroup of operators). We shall now give an example of such a linear dynamical system  $f(t)p$  for which  $\|f(t)\|$  has a

discontinuity at  $t = 0$ , and, all the more, there is no uniform continuity of  $f(t)$  at  $t = 0$ , since

$$\|f(t) - f(0)\| \geq \|\|f(t)\| - \|f(0)\|\|. \quad (2)$$

As the Banach space  $B$ , take the set of all real functions  $y(x)$  continuous on the closed interval  $[-\infty, \infty]$ , and define the operations in the natural way. The function  $y(x)$ , as a point of the space  $B$ , will be denoted by the letter  $p$ . Let  $g(x)$  be an arbitrary function defined on the whole number axis and bounded by positive numbers:  $0 < a \leq g(x) \leq A$ . If  $p = y(x)$ , then put  $\|p\| = \sup_{-\infty < x < \infty} |y(x)|g(x)$ .

It is easy to verify that the axioms of a norm are satisfied, and to establish the completeness of the space. Define in the space  $B$  the dynamical system  $f(t)p$ . If  $p = y(x)$ , then put  $f(t)p = y(x+t)$ . The fulfillment of the axioms  $A_1, A_2, A_4$  is obvious.

We shall now show that  $\lim_{t \rightarrow t_0} f(t)p = f(t_0)p$ . Take a fixed point  $p = y(x)$  and an arbitrary  $\varepsilon > 0$ . By the uniform continuity of the function  $y(x)$  on  $[-\infty, \infty]$ , there exists a number  $\delta > 0$  such that for  $|\Delta x| < \delta$  and any  $x \in (-\infty, \infty)$  the inequality  $|y(x + \Delta x) - y(x)| < \varepsilon/A$  holds. Let  $|t - t_0| < \delta$ . Then

$$\|f(t)p - f(t_0)p\| = \sup_{-\infty < x < \infty} |y(x+t) - y(x+t_0)|g(x) \leq \frac{\varepsilon}{A} \cdot A = \varepsilon.$$

For fixed  $t$  the operator  $f(t)$  is additive and homogeneous. Let us prove its boundedness. In our case  $\|p\| \leq 1$  means that  $\sup_{-\infty < x < \infty} |y(x)|g(x) \leq 1$ , and hence  $|y(x)| \leq 1/a$ . Consequently,

$$\|f(t)\| = \sup_{\|p\| \leq 1} \|f(t)p\| \leq \sup_{|y(x)| \leq 1/a} \sup_{-\infty < x < \infty} |y(x+t)|g(x) \leq A/a.$$

Thus the fulfillment of all conditions of Theorem 2 has been proved.

The properties of  $\|f(t)\|$  depend on the form of the function  $g(x)$ . For example, let  $g(x) = 1$  for  $x \leq 0$ ,  $g(x) = 1/2$  for  $x > 0$ ;  $\|f(t)\| = 1$  for  $t \leq 0$ ,  $\|f(t)\| = 2$  for  $t > 0$ . Then  $\|f(t)\|$  may also have a countable number of discontinuity points.

We shall show this. Let, for  $n-1 < x \leq n$  ( $n = 1, 2, \dots$ ),

$$g(x) = \frac{2^n + 1}{2^{n+1}},$$

and for  $x \leq 0$ ,  $g(x) = 1$ . It can be verified that for  $n-1 < t \leq n$  ( $n = 1, 2, \dots$ )

and for every  $x$

$$g(x-t) \leq \frac{2^{n+1}}{2^n + 1} g(x).$$

If  $n - 1 < t \leq n$  ( $n = 1, 2, \dots$ ), then we have

$$\begin{aligned} \|f(t)p\| &= \sup_{-\infty < x < \infty} |y(x+t)g(x)| = \sup_{-\infty < x < \infty} |y(x)|g(x-t) \leq \\ &\leq \sup_{-\infty < x < \infty} |y(x)|g(x) \frac{2^{n+1}}{2^n + 1} = \frac{2^{n+1}}{2^n + 1} |p|. \end{aligned}$$

Hence

$$\|f(t)\| \leq \frac{2^{n+1}}{2^n + 1}.$$

To establish the reverse inequality, for a given fixed  $t \in (n - 1, n]$  take  $p = y(x) = 1$  for  $x \leq n - 1$ ;  $p = \alpha x + \beta$  for  $n - 1 < x < (n - 1) + \frac{1}{2}[t - (n - 1)]$ ;  $p = 2^{n+1}/(2^n + 1)$  for  $x \geq \frac{1}{2}(n - 1 + t)$  (where  $\alpha$  and  $\beta$  are chosen so that  $y(x)$  is continuous). Obviously,  $\|p\| = 1$ . Now let  $x_0 = -\frac{1}{2}[t - (n - 1)]$ . Since  $x_0 + t = \frac{1}{2}(n - 1 + t)$ , we have  $y(x_0 + t) = 2^{n+1}/(2^n + 1)$ , while  $g(x_0) = 1$ , since  $x_0 < 0$ . Thus, for the indicated  $t$  and  $p$ ,

$$\|f(t)p\| = \sup_{-\infty < x < \infty} |y(x+t)g(x)| \geq y(x_0+t)g(x_0+t)g(x_0) = \frac{2^{n+1}}{2^n + 1}.$$

Thus, for  $n - 1 < t \leq n$ ,

$$\|f(t)\| = \frac{2^{n+1}}{2^n + 1} \quad (n = 1, 2, \dots).$$

For  $t \leq 0$ ,  $\|f(t)\| = 1$ , as is easy to verify. Consequently,  $\|f(t)\| = 1/g(x)$  and is discontinuous at the points  $t = 0, 1, 2, \dots$

It may be noted that even when  $g(x) \equiv 1$  we obtain a linear dynamical system which does not correspond to any differential equation (1), and in this case  $\|f(t)\| \equiv 1$ . For a finite-dimensional space the function  $\|f(t)\|$  is continuous, which follows from Theorem 3, the relation  $\|f(t) - f(t_0)\| \leq \|f(t_0)\| \cdot \|f(t - t_0) - I\|$ , and inequality (2) with  $f(0)$  replaced by  $f(t_0)$ .

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## CITED LITERATURE

1. M. G. Krein, *Uspekhi Mat. Nauk*, 3, no. 3(25), 166 (1948).
2. K. P. Persidskii, *Izv. AN KazSSR, Ser. Mat. i Mekh.*, No. 97, issue 4, 3 (1950).
3. K. P. Persidskii, *Izv. AN KazSSR, Ser. Astr., Fiz., Mat. i Mekh.*, No. 116, issue 1(6), 64 (1952).
4. V. V. Nemytskii, V. V. Stepanov, *Qualitative Theory of Differential Equations*, 1949.
5. E. Hille, *Functional Analysis and Semigroups*, IL, 1951.

*Note: Figure translations are in progress. See original paper for figures.*

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