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Abstract

Full Text

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MATRIX ELEMENTS OF IRREDUCIBLE UNITARY REPRESENTATIONS OF THE GROUP OF REAL ORTHOGONAL MATRICES AND OF THE MOTION GROUP OF $(n - 1)$ -DIMENSIONAL EUCLIDEAN SPACE

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In this note the matrix elements of irreducible unitary representations of the group $SO(n)$ of real orthogonal matrices (the rotation group of n -dimensional Euclidean space) and of the group $M(n-1)$ of motions of $(n-1)$ -dimensional Euclidean space are computed. Representations of the group $SO(n)$ were studied by É. Cartan ⁽¹⁾, and effective formulas for the infinitesimal operators of these representations were given by I. M. Gel' fand and M. L. Tsetlin ⁽²⁾. The matrix elements found in our note are new special functions, which in certain special cases reduce to Gegenbauer polynomials (for the group $SO(n)$) and Bessel functions (for the group $M(n-1)$). The case $n = 4$ is treated in ⁽³⁾, where these special functions are expressed in terms of known special functions.

In the group $SO(n)$ there exists a system of parameters analogous to the Euler angles for the group $SO(3)$. Each element ω of the group $SO(n)$ can be represented in the form

$$\omega = \psi \omega_{n-1} \omega_{n-2} \cdots \omega_1, \quad (1)$$

where ψ is a rotation leaving the axis x_n fixed (i.e., an element of the subgroup $SO(n-1)$), and ω_k is a rotation in the plane (x_k, x_{k+1}) through the angle $\varphi_k^{(n-1)}$. Decomposing in an analogous way the rotation ψ , etc., we obtain a system of parameters $\varphi_k^{(t)}$, $1 \leq t \leq n-1$, $1 \leq k \leq t$, which for $n = 3$ turns into the system of Euler angles. The invariant measure in these parameters is written as a product of invariant measures on spheres of dimensions $n-1, n-2, \dots, 1$. Multiplication of two elements of the group $SO(n)$, by virtue of the decomposition (1) and certain commutation relations, reduces to multiplications in the group $SO(n-1)$ and to replacing a product of the form $\omega_{n-1}^* \omega_{n-2} \omega_{n-1}^{**}$ by a product of the form $\omega_{n-2}^* \omega_{n-1} \omega_{n-2}^{**}$ ($\omega_k, \omega_k^*, \omega_k^{**}$ are rotations in the plane (x_k, x_{k+1})).

As shown in ⁽¹⁾, irreducible unitary representations of the group $SO(n)$ are specified by a set $\{L\}$ of numbers m_k , $1 \leq k \leq [n/2]$, $m_k \geq m_{k+1}$, all the numbers m_k being simultaneously integral or half-integral. The basis vectors ξ_α^L of such representations are numbered by means of the schemes described in ⁽²⁾ (formulas (2), (2')). Accordingly the matrix elements of the representation are written in the form $t_{\alpha\beta}^L(\omega)$, where α and β are schemes from ⁽²⁾.

We shall compute the matrix elements of irreducible unitary representations of the group $SO(n)$ by induction on n . For $n = 2$ these matrix elements are known; they are equal to $e^{ik\varphi}$. Suppose that the matrix elements are already known for all irreducible unitary representations of the groups $SO(k)$, $2 \leq k \leq n-1$. Specify numbers m_k , $1 \leq k \leq [n/2]$, determining a representation of the group $SO(n)$. Let α be one of the schemes corresponding

basis vectors of this representation. Then the first row A of this pattern determines the representation $\psi \rightarrow Q_\psi$ of the group $SO(n-1)$, while the pattern α , obtained from α by deleting the first row, is the basis vector ξ_α of the representation Q_ψ . We denote by $B(L, A)$ the set of all patterns corresponding to those basis vectors of the representation Q_ψ for which the first row consists of the numbers m_k , $2 \leq k \leq [n/2]$, and by $\xi_\alpha^L(\psi)$ the set of matrix elements of the representation Q_ψ having the form $t_{\nu\alpha}^A(\psi)$, $\nu \in B(L, A)$. Then $\xi_\alpha^L(\psi)$ may be regarded as a vector-function defined on the group $SO(n-1)$. Denote by $\bar{H}_L$ the finite-dimensional space spanned by all vector-functions $\xi_\alpha^L(\psi)$. The representation $\omega \rightarrow T_\omega$ of the group $SO(n)$, determined by the numbers m_k , $1 \leq k \leq [n/2]$, is constructed as follows. To an element ψ_0 of the subgroup $SO(n-1)$ there is assigned the operator T_{ψ_0} , mapping a vector-function $\xi(\psi)$ from $\bar{H}_L$ into the vector-function $\xi(\psi\psi_0)$. To the rotation ω_{n-1} through an angle λ in the plane (x_{n-1}, x_n) we assign the operator T_λ , mapping the vector-function $\xi(\psi)$ into the vector-function $T_\lambda\xi(\psi)$, which is obtained from $\xi(\psi)$ by replacing $\cos \theta$ by

$$\frac{\cos \lambda \cos \theta + i \sin \lambda}{i \sin \lambda \cos \theta + \cos \lambda} \quad (2)$$

and multiplying by $(i \sin \lambda \cos \theta + \cos \lambda)^{m_1}$. By θ we have denoted the angle $\varphi_{n-2}^{(n-2)}$ corresponding, in the parametrization described above, to the rotation ψ . For any element ω of $SO(n)$ we set, in accordance with the decomposition (1), $T_\omega = T_\psi T_\lambda T_{\psi_1}$, where ψ_1 denotes the element $\omega_{n-2}\omega_{n-1} \dots \omega_1$ of the subgroup $SO(n-1)$. It can be shown that the correspondence $\omega \rightarrow T_\omega$ is an irreducible representation of the group $SO(n)$. The scalar product with respect to which this representation is unitary is given by a formula of the form

$$(\xi_\alpha^L(\psi), \xi_\beta^L(\psi)) = 0, \quad \text{if } \alpha \neq \beta;$$

$$(\xi_\alpha^L(\psi), \xi_\alpha^L(\psi)) = C \int \sum_{\nu \in B(L, A)} |t_{\nu\alpha}^A(\psi)|^2 d\mu(\psi), \quad (3)$$

where C is a certain number depending on α , L , and n .

The matrix elements of the representations thus constructed are computed by the formula $t_{\alpha\beta}^L(\omega) = (T_\omega \xi_\beta^L, \xi_\alpha^L)$. Let ψ_0 be an element of the subgroup $SO(n-1)$. Then the corresponding operator T_{ψ_0} is given by a block-diagonal matrix on whose main diagonal stand the matrices of the irreducible unitary representations of the subgroup $SO(n-1)$ determined by the first rows of admissible patterns. Let us now consider the rotation ω_{n-1} through the angle λ in the plane (x_{n-1}, x_n) . Let $\|t_{\alpha\beta}^L(\lambda)\|$ be the matrix of the operator T_λ . Then $t_{\alpha\beta}^L(\lambda) = 0$ if the patterns $\bar{\alpha}$ and $\bar{\beta}$, obtained from α and β by deleting the first rows A and B , are different. Suppose now that $\bar{\alpha} = \bar{\beta}$. Denote by K the set of numbers m_k , $2 \leq k \leq [n/2]$, by N the first row of the pattern $\bar{\alpha} = \bar{\beta}$, by γ the pattern obtained from $\bar{\alpha}$ by deleting the first row, and by ν the pattern obtained from α by replacing the first row by K . By $P_{M,N,\gamma}^A(\cos \theta)$ denote the value of the matrix element $t_{\nu\alpha}^A(\psi)$ corresponding to the rotation through the angle θ in the plane (x_{n-2}, x_{n-1}) . Then

$$t_{\alpha\beta}^L(\lambda) = P_{A,B,\bar{\alpha}}^L(\cos \lambda) =$$

$$= C \int_{-1}^1 \left(P_{M,N,\gamma}^B \left(\frac{x \cos \lambda + i \sin \lambda}{ix \sin \lambda + \cos \lambda} \right) \right) P_{M,N,\gamma}^A(x) (ix \sin \lambda + \cos \lambda)^{m_1} (1-x^2)^{\frac{n-4}{2}} dx, \quad (4)$$

where C is a certain constant. If the patterns α and β consist of zeros, then $t_{\alpha\beta}^L(\lambda)$ coincides, up to a constant factor, with the Gegenbauer polynomial $C_{m_1}^p(\cos \lambda)$, $p = (n-2)/2$.

The function $P_{A,B,\bar{\alpha}}^L(\cos \theta)$ satisfies the following theorem.

Addition theorem. *Let the parameters $\theta_1, \varphi, \theta_2$ and $\varphi_1, \theta, \varphi_2$ be connected by the relations*

$$\cos \theta = \cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \cos \varphi, \quad (5)$$

$$\operatorname{tg} \varphi_1 = \frac{\sin \varphi \sin \theta_1}{\sin \theta_1 \cos \theta_2 \cos \varphi + \cos \theta_1 \sin \theta_2}, \quad (6)$$

$$\operatorname{tg} \varphi_2 = \frac{\sin \varphi \sin \theta_2}{\cos \theta_1 \sin \theta_2 \cos \varphi + \cos \theta_2 \sin \theta_1}. \quad (7)$$

Then the equality holds

$$\sum_{N_s} P_{N_1, N_s, \gamma}^A(\cos \varphi_1) P_{A, B, \beta_3}^L(\cos \theta) P_{N_s, N_2, \gamma}^B(\cos \varphi_2) =$$

$$= \sum_C P_{A,C,\beta_1}^L(\cos \theta_1) P_{N_1,N_2,\gamma}^C(\cos \varphi) P_{C,B,\beta_2}^L(\cos \theta_2), \quad (8)$$

where β_k denotes the scheme obtained from the scheme γ by adjoining, as the first row, the sequence N_k , $k = 1, 2, 3$.

From the above addition theorem there follow the infinitesimal relations between the basis vectors given in (2).

Representations of the group $M(n-1)$ are constructed in many respects analogously to representations of the group $SO(n)$. They are specified by a set $\{K\}$ of integral or half-integral numbers m_k , $2 \leq k \leq [n/2]$, $m_k \geq m_{k+1}$, and by a number $\rho > 0$. The schemes corresponding to the basis vectors of the representations are analogous to the schemes from (2), with the sole difference that the first element of the first row may take arbitrarily large values. Accordingly, the representation space $H_{K,\rho}$ is infinite-dimensional. To a rotation ψ_0 of the $(n-1)$ -dimensional space there corresponds an operator T_{ψ_0} , which takes the vector-function $\xi(\psi)$ into the vector-function $\xi(\psi\psi_0)$. To a translation by a vector of length R , directed along the axis x_{n-1} , there corresponds an operator T_R , which takes the vector-function $\xi(\psi)$ into the vector-function $e^{iR\rho \cos \theta} \xi(\psi)$. Here θ denotes the angle $\varphi_{n-2}^{(n-2)}$ corresponding to the element ψ of the group $SO(n-1)$ in the parametrization described above. The matrix of the operator T_{ψ_0} is an infinite block-diagonal matrix, on whose main diagonal stand the matrices of irreducible unitary representations of the group $SO(n-1)$. The matrix of the operator T_R consists of elements $t_{\alpha\beta}^{K,\rho}(R)$, where $t_{\alpha\beta}^{K,\rho}(R) = 0$ if $\bar{\alpha} \neq \bar{\beta}$; if $\bar{\alpha} = \bar{\beta}$, then $t_{\alpha\beta}^{K,\rho}(R) = J_{A,B,\bar{\alpha}}^K(\rho R)$, where

$$J_{A,B,\bar{\alpha}}^K(\mu) = C_1 \int_{-1}^1 e^{i\mu x} P_{M,N,\gamma}^A(x) P_{M,N,\gamma}^B(x) (1-x^2)^{(n-4)/2} dx \quad (9)$$

(the notation is the same as in formula (4)). In particular, if all the numbers in the schemes α and β are equal to zero, then

$$J_{A,B,\bar{\alpha}}^K(x) = C_2 \frac{J_p(x)}{x^p}, \quad p = \frac{n-3}{2}.$$

The functions $J_{A,B,\bar{\alpha}}^K(x)$ also satisfy an addition theorem, a special case of which (for $n=4$) was established in (3). The functions $J_{A,B,\bar{\alpha}}^K(x)$ may be regarded as limits of the functions $P_{A,B,\bar{\alpha}}^L(\cos \theta)$, when $\theta \rightarrow 0$ and $m_1 \rightarrow \infty$ in such a way that $\lim \theta m_1 = x$.

Functions analogous to the functions $P_{A,B,\bar{\alpha}}^L(\cos \theta)$ can be obtained by considering irreducible unitary representations of the group of linear transformations preserving the quadratic form

$$\sum_{k=1}^{n-1} x_k^2 - x_n^2.$$

In this case

instead of the fractional-linear substitution (2), one must consider the fractional-linear substitution

$$\frac{\operatorname{ch} \lambda \cos \theta + \operatorname{sh} \lambda}{\operatorname{sh} \lambda \cos \theta + \operatorname{ch} \lambda},$$

while m_1 assumes certain complex values. Correspondingly, the form of the addition theorem changes. In particular, by this method one computes the matrix elements of irreducible unitary representations of the Lorentz group.

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