

MATHEMATICS

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Abstract

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REDUCTION OF A LINEAR SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS TO A SYSTEM OF NORMAL TYPE

(Presented by Academician A. N. Kolmogorov on 22 I 1957)

Consider a system of N linear partial differential equations with constant coefficients of the form

$$\frac{\partial^{n_i} u_i(x_1, \dots, x_n, t)}{\partial t^{n_i}} = \sum_{j=1}^N \sum_{\langle m_0 \dots m_n \rangle} A_{m_0 \dots m_n}^{ij} \frac{\partial^{m_0 + \dots + m_n} u_j(x_1, \dots, x_n, t)}{\partial t^{m_0} \partial x_1^{m_1} \dots \partial x_n^{m_n}} \quad (1)$$

$$(i = 1, \dots, N);$$

here the summation is carried out over all possible sets of indices $\langle m_0 \dots m_n \rangle$, where $m_0 < n_j$; $m_i < M$ ($i = 1, \dots, n$).

Let $\lambda_1(s), \dots, \lambda_{N_1}(s)$ ($N_1 = \sum_{i=1}^N n_i$, $s = (s_1, \dots, s_n)$) be the characteristic roots of the system (1), i.e. the roots of the polynomial

$$\lambda^{N_1} - P_1(s)\lambda^{N_1-1} - \dots - P_{N_1}(s) \equiv$$

$$\equiv \det \left(\left\| \begin{array}{cccc} \lambda^{n_1} & 0 & \dots & 0 \\ 0 & \lambda^{n_2} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & \lambda^{n_N} \end{array} \right\| - \left\| \sum_{\langle m_0 \dots m_n \rangle} A_{m_0 \dots m_n}^{ij} \lambda^{m_0} (is_1)^{m_1} \dots (is_n)^{m_n} \right\| \right), \quad (2)$$

where $P_i(s)$ are polynomials with respect to $s_1, \dots, s_n$, and suppose that, for

$$|s| = \sqrt{|s_1|^2 + \dots + |s_n|^2} \gg 1,$$

$$\max_{1 \leq j \leq N_1} \operatorname{Re} \lambda_j(s) \leq C_k |s|^k. \quad (3)$$

The **reduced order** of the system (1) is the number

$$p_0 = \inf k; \quad (4)$$

the lower bound is taken over all k for which (3) holds. The number p_0 plays an important role in questions of uniqueness and correctness of the solution of the Cauchy problem for the system (1) ^(1,2).

Theorem 1. *The reduced order p_0 of the system (1) can be found from the formula*

$$p_0 = \max_{1 \leq i \leq N_1} \frac{p_i}{i}, \quad (5)$$

where p_i is the highest degree, with respect to the set of variables $s_1, \dots, s_n$, of the polynomial $P_i(s)$ in the identity (2).

Proof. As shown in (3), it follows from (3) that for $|s| \geq 1$

$$\max_{1 \leq j \leq N_1} |\lambda_j(s)| \leq C_{1k} |s|^k. \quad (6)$$

Since

$$P_r(s) = \sum_{\langle i_1 \dots i_r \rangle} \lambda_{i_1}(s) \dots \lambda_{i_r}(s) \quad (1 \leq r \leq N_1),$$

where the summation extends over all possible sets of r indices ($1 \leq i_k \leq N_1$, $1 \leq k \leq r$), it follows from (4) and (6) that, for $|s| \geq 1$ and for any $\varepsilon > 0$,

$$|P_r(s)| \leq A_{r,\varepsilon} |s|^{r(p_0 + \varepsilon)}.$$

Hence $p_r \leq r(p_0 + \varepsilon)$ ($1 \leq r \leq N_1$) for any $\varepsilon > 0$; consequently,

$$p_0 \geq \max_{1 \leq r \leq N_1} \frac{p_r}{r}. \quad (7)$$

Suppose that

$$p_0 > \max_{1 \leq r \leq N_1} \frac{p_r}{r}.$$

By virtue of (4) and (6), for the given $\varepsilon > 0$ one can find a number j_0 and a sequence of points $s_m = (s_{1m}, \dots, s_{nm})$ ($|s_m| \xrightarrow{m \rightarrow \infty} \infty$) such that

$$|\lambda_{j_0}(s_m)| > m|s_m|^{p_0 - \varepsilon}. \quad (8)$$

On the other hand,

$$\lambda_{j_0}^{N_1}(s) - P_1(s)\lambda_{j_0}^{N_1-1}(s) - \dots - P_{N_1}(s) \equiv 0, \quad (9)$$

but on the sequence s_m we have:

$$\begin{aligned} & |\lambda_{j_0}^{N_1}(s_m) - \dots - P_{N_1}(s_m)| = \\ & = |\lambda_{j_0}^{N_1}(s_m)| \left[1 - \frac{P_1(s_m)}{\lambda_{j_0}(s_m)} - \dots - \frac{P_{N_1}(s_m)}{\lambda_{j_0}^{N_1}(s_m)} \right]. \end{aligned} \quad (10)$$

The expression in square brackets tends to 1 as $m \rightarrow \infty$. Indeed, by (8),

$$\left| \frac{P_i(s_m)}{\lambda_{j_0}^i(s_m)} \right| \leq \frac{A|s_m|^{p_i}}{m^i|s_m|^{ip_0 - i\varepsilon}} = \frac{A}{m^i}|s_m|^{p_i - ip_0 + i\varepsilon} \xrightarrow{m \rightarrow \infty} 0 \quad (i = 1, \dots, N_1),$$

if the number $\varepsilon > 0$ is chosen so that, for all $i = 1, \dots, N_1$, the inequality $\varepsilon < p_0 - p_i/i$ holds.

Therefore, as follows from (8) and (10), the identity (9) cannot hold for $s = s_m$ with sufficiently large indices m , whence it follows that

$$p_0 \leq \max_{1 \leq r \leq N_1} \frac{p_r}{r}. \quad (11)$$

From (7) and (11) follows (5). The theorem is proved.

As follows from formula (5), p_0 is always a rational number. Any rational number can serve as the reduced order of some system of the form (1) ⁽²⁾.

We shall say that a system (1) with reduced order $p_0 = m/k$ (the fraction m/k is irreducible) has **normal type** if: 1) all

$$m_0 = 0; \quad 2) \quad n_i \leq k; \quad 3) \quad \sum_{i=1}^n m_i \leq m.$$

Theorem 2. *The Cauchy problem for every system of the form (1) can be reduced to the Cauchy problem for a system of normal type.*

Proof. By using the obvious notation of the derivatives of the unknown functions with respect to (time) t as new unknown functions, one can reduce system (1) to a form in which all $n_i = 1$ and all $m_0 = 0$, and then to a system of the form

$$\frac{\partial^{k_j} u_j(x_1, \dots, x_n, t)}{\partial t^{k_j}} = \sum_{l=1}^{k_j} P_{lj} \left(i \frac{\partial}{\partial x} \right) \frac{\partial^{k_j-l} u_j(x_1, \dots, x_n, t)}{\partial t^{k_j-l}} \quad (12)$$

$\left(j = 1, \dots, p; 1 \leq p \leq N_1; \sum_{j=1}^p k_j = N_1; P_{lj} \left(i \frac{\partial}{\partial x} \right) \text{ is a polynomial in } i \frac{\partial}{\partial x_1}, \dots, i \frac{\partial}{\partial x_n} \right.$

with constant coefficients), i.e., to one equation ($p = 1$) or to several equations, each of which is integrated independently of the others (3).

The Cauchy problem for system (1) is equivalent to the Cauchy problem for system (12). The characteristic roots of system (1) do not change under the transition to system (12); therefore both of these systems have the same reduced order p_0 . Suppose that $p_0 = m/k$ and that the fraction m/k is irreducible.

We shall reduce system (12) to a system of normal type as follows: denote ($j = 1, \dots, p, x = (x_1, \dots, x_n)$)

$$\frac{\partial u_j(x, t)}{\partial t} = u_{j1}(x, t), \dots, \frac{\partial^{k_j-1} u_j(x, t)}{\partial t^{k_j-1}} = u_{jk_j-1}(x, t);$$

$$\frac{\partial^m u_{jk_j-2}(x, t)}{\partial x_1^{r_1^1} \dots \partial x_n^{r_n^1}} = u_{jk_j-2}^{(r_1^1 \dots r_n^1)}(x, t) \quad \left(\sum_{i=1}^n r_i^1 = m \right); \quad \frac{\partial^m u_{jk_j-3}(x, t)}{\partial x_1^{r_1^1} \dots \partial x_n^{r_n^1}} = u_{jk_j-3}^{(r_1^1 \dots r_n^1)}(x, t);$$

$$\frac{\partial^{2m} u_{jk_j-3}(x, t)}{\partial x_1^{r_1^1+r_1^2} \dots \partial x_n^{r_n^1+r_n^2}} = u_{jk_j-3}^{(r_1^1 \dots r_n^1; r_1^2 \dots r_n^2)}(x, t) \quad \left(\sum_{i=1}^n r_i^1 = m, \sum_{i=1}^n r_i^2 = m \right);$$

...

$$\frac{\partial^m u_j(x, t)}{\partial x_1^{r_1^1} \dots \partial x_n^{r_n^1}} = u_j^{(r_1^1 \dots r_n^1)}(x, t);$$

...

$$\frac{\partial^{(k_j-1)m} u_j(x, t)}{\partial x_1^{r_1^1 + \dots + r_1^{k_j-1}} \dots \partial x_n^{r_n^1 + \dots + r_n^{k_j-1}}} = u_j^{\langle r_1^1 \dots r_n^1; \dots; r_1^{k_j-1} \dots r_n^{k_j-1} \rangle}(x, t)$$

$$\left(\sum_{i=1}^n r_i^q = m, \quad q = 1, 2, \dots, k_j - 1 \right).$$

Taking into account that the degree p_{lj} of the polynomial $P_{lj}(s)$ in the aggregate of variables $s_1, \dots, s_n$, by Theorem 1, satisfies the inequality $p_{lj} \leq l \frac{m}{k} \leq lm$, we can write

$$P_{lj} \left(i \frac{\partial}{\partial x_1}, \dots, i \frac{\partial}{\partial x_n} \right) = P_{lj}^{(1)} \left(i \frac{\partial}{\partial x} \right) + \sum_{\langle r_1^1 \dots r_n^1 \rangle} \frac{\partial^m}{\partial x_1^{r_1^1} \dots \partial x_n^{r_n^1}} P_{lj}^{\langle r_1^1 \dots r_n^1 \rangle} \left(i \frac{\partial}{\partial x} \right) + \dots$$

$$\dots + \sum_{\sum_{k=1}^{l-1} \sum_{i=1}^n r_i^k = m(l-1)} \frac{\partial^{m(l-1)}}{\partial x_1^{r_1^1 + \dots + r_1^{l-1}} \dots \partial x_n^{r_n^1 + \dots + r_n^{l-1}}} P_{lj}^{\langle r_1^1 \dots r_n^1; \dots; r_1^{l-1} \dots r_n^{l-1} \rangle} \left(i \frac{\partial}{\partial x} \right);$$

here the degree of the polynomial $P_{lj}^{\langle r_1^1 \dots r_n^1; \dots; r_1^L \dots r_n^L \rangle}(s)$ ($1 \leq L \leq l-1$) in the aggregate of the variables $s_1, \dots, s_n$ does not exceed m . Then system (12) can be written in the form $(u_{j_0}(x, t) \equiv u_j(x, t))$

$$\frac{\partial u_{jr}(x, t)}{\partial t} = u_{jr+1}(x, t) \quad (r = 0, \dots, k_j - k - 1);$$

$$\frac{\partial^h u_{jk_j-h}}{\partial t^h} = \sum_{l=1}^{k_j} \left[P_{lj}^1 \left(i \frac{\partial}{\partial x} \right) u_{jk_j-l}(x, t) + \dots \right.$$

$$\left. \dots + \sum P_{lj}^{\langle r_1^1 \dots r_n^1; \dots; r_1^{l-1} \dots r_n^{l-1} \rangle} \left(i \frac{\partial}{\partial x} \right) u_{jk_j-l}^{\langle r_1^1 \dots r_n^1; \dots; r_1^{l-1} \dots r_n^{l-1} \rangle}(x, t) \right]$$

$$(h = 1, \dots, k); \quad (13)$$

$$\frac{\partial u_{jk_j-\chi}^{\langle r_1^1 \dots r_n^1; \dots; r_1^q \dots r_n^q \rangle}(x, t)}{\partial t} = \frac{\partial^m u_{jk_j-\chi+1}^{\langle r_1^1 \dots r_n^1; \dots; r_1^{q-1} \dots r_n^{q-1} \rangle}(x, t)}{\partial x_1^{r_1^q} \dots \partial x_n^{r_n^q}}$$

$$\left(\sum_{j=1}^n r_j^\alpha = m; \quad \alpha = 1, \dots, q; \quad 2 \leq q+1 \leq \chi \leq k_j \right), \quad j = 1, \dots, p.$$

In order to see that system (13) is of normal type, it is necessary to show that its reduced order is equal to m/k , since properties 1), 2), and 3) for it in this case are evidently fulfilled.

The passage from system (12) to system (13) can be carried out successively by means of “elementary reductions,” consisting in the introduction of a new function $v_k = \partial u_k / \partial t$, or of a new function $v_{k,j} = \partial u_k / \partial x_j$. It is easy to verify that, under an elementary reduction which brings a system of the form (1) (and systems (12) and (13) have precisely this form) to a system of the same form, all characteristic roots of the system are preserved and only zero roots may be added. Consequently, the characteristic roots of system (13) differ from the characteristic roots of system (12) only by zero roots, and therefore systems (12) and (13) have the same reduced order $p_0 = m/k$. Thus, system (13) is of normal type.

In order that the Cauchy problem for system (13) be equivalent to the Cauchy problem for system (12) (and, consequently, also for system (1)), one must impose additional initial conditions—quite analogously to how this is done in reducing the Cauchy problem for a Kovalevskaya-type system to the Cauchy problem for the first-order system (4). The theorem is proved.

Corollary 1. If for system (1) $p_0 = 0$, then it can be reduced to a system of ordinary differential equations.

Corollary 2. If for system (1) $p_0 = 1$, then such a system can be reduced to a first-order system. This fact holds, for example, for hyperbolic systems ⁽³⁾.

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Note: Figure translations are in progress. See original paper for figures.

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